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COUNTEREXAMPLES TO THE BAUM–CONNES CONJECTURE

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The Baum–Connes conjecture [BC], [BCH] proposes a formula for the operator K-theory of reduced group C^*-algebras and foliation C^*-algebras. If G is the fundamental group of a finite CW-complex then the Baum–Connes conjecture for G can be viewed as an analytic counterpart of the Borel conjecture in manifold theory, which proposes a homological formula for the L-theory of the group ring $\mathbb{Z}[G]$. Moreover the Baum–Connes conjecture for a group G actually *implies* Novikov's conjecture that the higher signatures of a closed, oriented manifold with fundamental group G are oriented homotopy invariants. For this reason manifold theory has been a driving force behind work on the Baum–Connes conjecture, and in return operator K-theory techniques have proved some of the best known results on the homotopy invariance of higher signatures.

From the very beginning, generalizations to group actions have played an important role in the development of the Baum–Connes conjecture. More recently, further extensions have been proposed to general locally compact groupoids [T1] and to coarse geometric spaces [HiR], [R]. Current operator algebraic approaches to the Novikov conjecture rely quite heavily on these (see for instance [Hi]).

The Baum–Connes conjecture and its generalizations have now been verified in a variety of cases. For recent work on groups and group actions see [HiK], [L]; for work on groupoids see [T1]; and for work on coarse geometric spaces see [Y]. Indeed the scope of what has now been proved is quite remarkable, especially given the scant information which Baum and Connes had available to them at the outset. Of course the general Baum–Connes conjecture is broader still, applying as it does to *every* (second countable) locally compact groupoid, or even, in the case of the conjecture 'with coefficients', to every action of a such a groupoid on a C^*-algebra. The conjecture has fascinating points of contact not only with the Novikov conjecture but with Riemannian geometry, the representation theory of real and p-adic groups, and the spectral theory of discrete groups.

The conjecture asserts that a certain *Baum–Connes assembly map*

$$\mu\colon K_*^{\mathrm{top}}(G) \to K_*(C_r^*(G))$$

is an isomorphism of abelian groups, where $C_r^*(G)$ denotes the reduced C^*-algebra of the group or groupoid G. In the case of the conjecture with coefficients, the conjecture asserts that for every G-C^*-algebra A a certain generalized assembly map

$$\mu\colon K_*^{\mathrm{top}}(G,A) \to K_*(A \rtimes_r G)$$

is an isomorphism, where $A \rtimes_r G$ denotes the reduced crossed product C^*-algebra. The purpose of this note is to present counterexamples to:

- the injectivity and the surjectivity of the Baum–Connes map for Hausdorff groupoids;
- the injectivity and surjectivity of the Baum–Connes map for (non-Hausdorff) holonomy groupoids of foliations;
- the surjectivity of the Baum–Connes map for coarse geometric spaces;
- the surjectivity of the Baum–Connes map for discrete group actions on commutative C^*-algebras, contingent on certain as yet unpublished results of Gromov.

We have not obtained counterexamples to the Baum–Connes conjecture for groups alone (as opposed to group actions) or to the conjecture for foliations with Hausdorff holonomy groupoids, but in the light of what follows it is difficult to be confident that no such counterexample will be found soon.

All of our examples are based on essentially the same phenomenon: a failure of 'exactness' when various short exact sequences of algebras are completed to reduced C^*-algebras. This phenomenon was previously quite well known, at least in the context of groupoids. Our main observation – which is really rather simple – is that the failure of exactness can be detected at the level of K-theory. Since the various generalizations of the Baum–Connes conjecture predict exactness at the level of K-theory, we thereby obtain our counterexamples.

Our investigation actually began with the Baum–Connes conjecture for coarse geometric spaces, and was inspired by the following very elegant observation of Gromov (see the final section of [G1]). Guoliang Yu [Y] proved that the coarse Baum–Connes conjecture is true for any bounded geometry metric space which admits a uniform embedding into Hilbert space. At the time Yu announced his result there were no known examples of bounded geometry spaces which did *not* admit such an embedding, but Gromov pointed out that no expanding sequence of graphs so embeds. And indeed it turns out to be rather simple to construct a counterexample to the coarse Baum–Connes conjecture starting from a suitable expanding sequence.

Gromov has taken his observation concerning expanders considerably further: by mapping expanding sequences of graphs into suitable Cayley graphs he has constructed by probabilistic means examples of finitely generated groups, even geometrically finite groups, which do not embed uniformly into Hilbert space. This answers a question raised by Gromov himself (see [G2, p. 218], and [FRR, p. 67, problem 4]). Combining Gromov's examples with some simple observations about the C^*-algebras of coarse-geometric spaces we are able to present examples of group actions on compact spaces for which the Baum–Connes conjecture fails. In other words, contingent on Gromov's results, there exist countable groups G and compact, metrizable G-spaces X such that the Baum–Connes conjecture for G, with coefficients in the G-C^*-algebra $C(X)$, is false.

Since the above examples are derived from a failure of exactness, we are able to conclude that for the Gromov's groups G the reduced C^*-algebras $C^*_r(G)$ are not exact. This settles an old question in abstract C^*-algebra theory. The same result has been obtained, in a different way, by Anantharaman-Delaroche [A], Guentner and Kaminker [GuK] and Ozawa [O]. Their work provides a separate means of deducing from Gromov's constructions that there exist discrete groups for which $C^*_r(G)$ is not an exact C^*-algebra.

It is worth noting that our method does not give counter-examples to the Baum–Connes conjecture if we replace reduced C^*-algebras by L^1-algebras (the 'generalized Bost conjecture') or by any unconditional completion (as discussed in [L]). In most examples, one can in fact show that the Baum–Connes maps for unconditional completions remain isomorphisms. Furthermore we have not obtained counterexamples to the injectivity of the 'full' Baum–Connes map (which involves full, as opposed to reduced, group and groupoid C^*-algebras).

1 The Main Idea

Let G be a locally compact groupoid with Haar system. The Baum–Connes conjecture, as generalized by Tu [T1,2], proposes a formula for the K-theory of the reduced C^*-algebra of G. We refer the reader to Tu's papers for details, as well as to Le Gall's papers [Le1,2], for further information on operator K-theory for groupoids. One of the interesting features of our counterexamples is that they rely on very little background information concerning the formulation of the conjecture: our examples contradict not only the Baum–Connes conjecture but also all possible variants of the Baum–Connes conjecture which satisfy only a small list of axioms.

In the Baum–Connes theory one associates *topological K-theory groups* $K_*^{\mathrm{top}}(G)$ to each groupoid G as above. More generally to each action of G on a C^*-algebra A one associates topological K-theory groups $K_*^{\mathrm{top}}(G, A)$. For example, if G is the fundamental group of a closed, aspherical, even-dimensional spin^c-manifold M then $K_*^{\mathrm{top}}(G)$ identifies with the topological K-theory of M, while $K_*^{\mathrm{top}}(G, A)$ identifies with the C^*-algebra K-theory of the sections of the flat C^*-algebra bundle over M which is associated to the given action of G on A. The groups $K_*^{\mathrm{top}}(G, A)$ are functorial in A with respect to G-equivariant $*$-homomorphisms.

In what follows we shall be mostly concerned with an important special case of this functoriality. Let G^0 denote the space of objects of the groupoid G. Then the commutative C^*-algebra $C_0(G^0)$ is in a natural way a G-C^*-algebra. If F is a closed subset of G^0 which is *saturated*, meaning that every morphism whose source belongs to F also has its range in F, then $C_0(F)$ is a G-C^*-algebra quotient of $C_0(G^0)$. Now the topological K-theory groups $K_*^{\mathrm{top}}(G, C_0(G^0))$ identify with $K_*^{\mathrm{top}}(G)$, while the topological K-theory groups $K_*^{\mathrm{top}}(G, C_0(F))$ identify with $K_*^{\mathrm{top}}(G_F)$, where G_F is the closed subgroupoid of G comprised of morphisms whose source and range belong to F. There is therefore a natural *restriction morphism*

$$K_*^{\mathrm{top}}(G) \to K_*^{\mathrm{top}}(G_F)$$

associated to every closed, saturated subset F of G^0.

Now let $A \rtimes_{\max} G$ and $A \rtimes_r G$ be the full and reduced crossed product algebras, respectively. These too are functorial in A. If $A = C_0(G^0)$ then the crossed products are the full and reduced groupoid algebras $C^*_{\max}(G)$ and $C^*_r(G)$, while if $A = C_0(F)$ then they are the full and reduced algebras of the subgroupoid G_F. The associated restriction $*$-homomorphisms

$$C^*_{\max}(G) \to C^*_{\max}(G_F) \quad \text{and} \quad C^*_r(G) \to C^*_r(G_F)$$

are nothing more than actual restriction from the dense subalgebra $C_c(G)$ contained within the groupoid C^*-algebras for G to the algebra $C_c(G_F)$ contained within the groupoid C^*-algebras for G_F.

As far as this note is concerned, the central object in the Baum–Connes theory is the *full assembly map*

$$\mu_{\max} \colon K_*^{\mathrm{top}}(G, A) \to K_*(A \rtimes_{\max} G) .$$

It is functorial in A for G-equivariant $*$-homomorphisms. In particular, if F is a closed, saturated subset of G^0 then there is a commuting *restriction diagram*

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G) & \rightarrow & K_*^{\mathrm{top}}(G_F) \\ \downarrow & & \downarrow \\ K_*(C^*_{\max}(G)) & \rightarrow & K_*(C^*_{\max}(G_F)) \end{array}$$

Until we discuss injectivity counterexamples in section 3 we shall need no further information about topological K-theory and the full assembly map.

The Baum–Connes conjecture is that the *reduced assembly map*

$$\mu\colon K_*^{\mathrm{top}}(G,A) \to K_*(A \rtimes_r G)\,,$$

which is the composition of the full assembly map with the K-theory map associated to the canonical surjection from $A \rtimes_{\max} G$ onto $A \rtimes_r G$, is an *isomorphism.*

From now on the term *Baum–Connes map* will refer to this reduced assembly map; we shall not need to concern ourselves directly with the full assembly map, except to keep in mind that the Baum–Connes map factors through it.

There is a restriction diagram for the Baum–Connes map,

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G) & \rightarrow & K_*^{\mathrm{top}}(G_F) \\ \downarrow & & \downarrow \\ K_*(C^*_r(G)) & \rightarrow & K_*(C^*_r(G_F)) \end{array}$$

whose commutativity follows immediately from the same for the full assembly map.

The main idea in all our counterexamples is to exploit the compatibility of full and reduced assembly with restriction in the following way. Let F be a closed, saturated subset of G^0 and let U be its complement in G^0. Let G_F be the groupoid associated to F, as above, and let G_U be its complement in G, which is the open subgroupoid comprised of all morphisms with source and range belonging to U. Associated to the decomposition of G into G_F and G_U is a short exact sequence of full groupoid C^*-algebras

$$0 \to C^*_{\max}(G_U) \to C^*_{\max}(G) \to C^*_{\max}(G_F) \to 0\,,$$

in which the quotient map is restriction, as above, and the inclusion is the natural one induced from the inclusion of $C_c(G_U)$ into $C_c(G)$. The corresponding sequence for the reduced C^*-algebras,

$$0 \to C^*_r(G_U) \to C^*_r(G) \to C^*_r(G_F) \to 0$$

need *not* be exact at its middle term. Indeed this failure of exactness can even be detected at the level of K-theory: it is possible to construct examples in which the sequence

$$K_0(C_r^*(G_U)) \to K_0(C_r^*(G)) \to K_0(C_r^*(G_F)) \qquad (\star)$$

fails to be exact at its middle term (recall that K-theory is a half-exact functor – see for example [Bl] – so this failure of exactness at the level of K-theory certainly implies a failure of exactness at the level of C^*-algebras). Constructions of this sort are precisely what we shall carry out in the subsequent sections of the paper. In this section we merely observe that every such construction immediately provides a counterexample to the Baum–Connes conjecture:

LEMMA 1. *Assume that the sequence* $(\star)$ *is not exact at its middle term.*

1. *If the Baum–Connes map* $K_0^{\mathrm{top}}(G_F) \to K_0(C_r^*(G_F))$ *is injective, then the Baum–Connes map* $K_0^{\mathrm{top}}(G) \to K_0(C_r^*(G))$ *fails to be surjective.*
2. *If the map* $K_0(C_{\max}^*(G_F)) \to K_0(C_r^*(G_F))$ *is injective, then the map* $K_0(C_{\max}^*(G)) \to K_0(C_r^*(G))$ *fails to be surjective and* a fortiori *the Baum–Connes map* $K_0^{\mathrm{top}}(G) \to K_0(C_r^*(G))$ *fails to be surjective.*

Proof. We need only chase around the following diagram,

$$\begin{array}{ccccc} K_0(C_r^*(G_U)) & \to & K_0(C_r^*(G)) & \to & K_0(C_r^*(G_F)) \\ \uparrow & & \uparrow & & \uparrow \\ K_0(C_{\max}^*(G_U)) & \to & K_0(C_{\max}^*(G)) & \to & K_0(C_{\max}^*(G_F)) \\ & & \uparrow & & \uparrow \\ & & K_0^{\mathrm{top}}(G) & \to & K_0^{\mathrm{top}}(G_F)\,, \end{array}$$

using the facts that the squares commute and that the middle line is exact at its middle term. To prove the first part of the lemma, let x be an element in $K_0(C_r^*(G))$ whose image is 0 in $K_0(C_r^*(G_F))$, but which is not in the image of $K_0(C_r^*(G_U))$. Then x is not in the range of the Baum–Connes map. Indeed, assume for the sake of a contradiction that x is the image of $y \in K_0^{\mathrm{top}}(G)$. Since the image of x in $K_0(C_r^*(G_F))$ is 0, it follows from our assumption of injectivity that the image of y in $K_0^{\mathrm{top}}(G_F)$ is 0. Therefore the image of y in $K_0(C_{\max}^*(G))$ lies in the image of $K_0(C_{\max}^*(G_U))$ and x lies in the image of $K_0(C_r^*(G_U))$ which leads to a contradiction, as required. The proof of the second part of the lemma is similar. □

In most of the cases that follow the Baum–Connes map for G_F will be injective. Therefore we shall find that the Baum–Connes map for G fails to be surjective. This failure of surjectivity can be made more starkly evident by a simple construction involving mapping cones. Suppose we are presented with a sequence of C^*-algebras and $*$-homomorphisms

$$J \xrightarrow{\alpha} A \xrightarrow{\beta} B$$

for which the composition of α and β is zero. Recall that the *mapping cone* of the $*$-homomorphism $\beta\colon A \to B$ is the C^*-algebra

$$C(\beta) = \big\{ a \oplus f \in A \oplus C[0,1] \otimes B : \beta(a) = f(0) \quad \text{and} \quad f(1) = 0 \big\} .$$

There is an obvious $*$-homomorphism γ from J into $C(\beta)$ and if the map induced by γ on K-theory is anisomorphism then the sequence

$$K_*(J) \to K_*(A) \to K_*(B) \qquad (\star\star)$$

is exact at its middle term. Let us consider this natural map $\gamma\colon J \to C(\beta)$ and construct the mapping cone of *it*. The K-theory of $C(\gamma)$ vanishes if and only if the $*$-homomorphism γ induces an isomorphism at the level of K-theory. In that case $(\star\star)$ is exact at its middle term. Hence if the sequence $(\star\star)$ *fails* to be exact at its middle term then the K-theory of $C(\gamma)$ is *non-zero*. If we start with a short exact sequence

$$0 \to J \to A \to B \to 0 ,$$

then the K-theory groups of $C(\gamma)$ vanish. Let us apply these observations to the short exact sequences of C^*-algebras associated to the decomposition $G = G_F \cup G_U$ that we discussed above. The associated sequence of full groupoid C^*-algebras is exact, and so the K-theory groups of the corresponding 'full' C^*-algebra $C(\gamma)$ vanish. On the other hand if we assume that the sequence $(\star)$ fails to be exact at its middle term then the K-theory groups of the corresponding 'reduced' C^*-algebra $C(\gamma)$ do *not* vanish. Now in both cases, the mapping cone $C(\gamma)$ is a groupoid C^*-algebra – a full groupoid C^*-algebra in the first case and a reduced in the second. If S denotes the space obtained from a closed square by removing two adjacent sides (thus S is a product of $[0,1[$ with itself), and if $E_1, E_2 \subseteq S$ are the two remaining sides, then the groupoid in question is

$$\mathcal{G}(F) = G_F \times (S \setminus E_1) \cup G_U \times E_2 .$$

(This is an open subgroupoid of a closed subgroupoid of $G \times S$. The latter should be viewed here as a constant family of groupoids over S, and therefore as a groupoid in its own right. In this way $\mathcal{G}(F)$ carries a locally compact groupoid topology.) It follows from our general discussion of mapping cones that

- $K_*(C^*_{\max}(\mathcal{G}(F))) = 0$ for every G and every closed saturated set $F \subseteq G^0$.
- $K_*(C^*_r(\mathcal{G}(F))) \neq 0$ if the K-theory sequence $(\star)$ associated to the closed saturated set $F \subseteq G^0$ fails to be exact at its middle term.

Since the Baum–Connes assembly map for the groupoid $\mathcal{G}(F)$ factors through the K-theory of the full groupoid C^*-algebra we obtain the following result:

PROPOSITION 2. *Let G be a locally compact groupoid with a Haar system. If the K-theory sequence $(\star)$ associated to a closed saturated set $F \subseteq G^0$ fails to be exact at its middle term then the Baum–Connes assembly map for the associated groupoid $\mathcal{G}(F)$ is zero, while the K-theory of the reduced C^*-algebra for $\mathcal{G}(F)$ is non-zero. In particular, if the sequence $(\star)$ fails to be exact at its middle term then the Baum–Connes assembly map for $\mathcal{G}(F)$ fails to be surjective.* □

We remark that in fact the topological K-theory of $\mathcal{G}(F)$ is zero, so that the Baum–Connes map is for trivial reasons injective in these examples.

2 Surjectivity Counterexamples

Let Γ_∞ be an infinite discrete group. Let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a family of finite groups and let $\{\pi_n \colon \Gamma_\infty \to \Gamma_n\}_{n\in\mathbb{N}}$ be a family of surjective homomorphisms. For convenience, let us denote by $\pi_\infty \colon \Gamma_\infty \to \Gamma_\infty$ the identity map from Γ_∞ to itself.

Define a groupoid G in the following way. Let G^0 be $\overline{\mathbb{N}}$, the one-point compactification of the natural numbers, and let the space of morphisms in G be the quotient of $\Gamma_\infty \times \overline{\mathbb{N}}$ by the following equivalence relation: $(g,n) \sim (h,m)$ if and only if $n = m \in \overline{\mathbb{N}}$ and $\pi_n(g) = \pi_n(h)$. Then G is a groupoid and in fact a locally compact groupoid in the quotient topology. It is of course just a continuous family of groups over $\overline{\mathbb{N}}$, whose fiber over n is Γ_n. The groupoid is Hausdorff if and only if, for every $g \in \Gamma_\infty \setminus \{1\}$, the set $\{n \in \mathbb{N}, \pi_n(g) = 1\}$ is finite.

We recall that in the present situation the C^*-algebra $C^*_r(G)$ is the completion of $C_c(\Gamma_\infty \times \overline{\mathbb{N}})$ for the following C^*-seminorm:

$$\|f\|_{C^*_r(G)} = \sup_{n\in\overline{\mathbb{N}}} \|\lambda_n(f_n)\|_{\mathcal{L}(\ell^2(\Gamma_n))} .$$

Here $f_n \colon \Gamma_\infty \to \mathbb{C}$ is the restriction of f to $\Gamma_\infty \cong \Gamma_\infty \times \{n\}$ and λ_n is the composition of the surjection from Γ_∞ to Γ_n with the regular representation of Γ_n.

Consider the closed saturated set $\{\infty\}$ of $G^0 = \overline{\mathbb{N}}$. By abuse of notation, we write G_∞ instead of $G_{\{\infty\}} \cong \Gamma_\infty$. Let us form the following particular instance of the sequence $(\star)$ that was analyzed in the previous section:

$$K_0(C^*_r(G_{\mathbb{N}})) \to K_0(C^*_r(G)) \to K_0(C^*_r(G_\infty)) . \qquad (\star\star\star)$$

PROPOSITION 3. *Assume that the trivial representation of Γ_∞ is isolated in the direct sum unitary representation $\oplus_{n\in\overline{\mathbb{N}}}\lambda_n$ of Γ_∞. Then the sequence $(\star\star\star)$ fails to be exact at its middle term.*

Proof. Each group element $g \in \Gamma_\infty$ determines a canonicalelement in $C_c(\Gamma_\infty \times \overline{\mathbb{N}})$, namely the characteristic function of $\{g\} \times \overline{\mathbb{N}}$, and in this waya $*$-homomorphism $\pi\colon \mathbb{C}[\Gamma_\infty] \to C_c(G)$ is determined. It follows from the above definition of the norm in $C^*_r(G)$ that if $C^*_\pi(\Gamma_\infty)$ denotes the completion of $\mathbb{C}[\Gamma_\infty]$ in the norm associated to the representation $\oplus_{n\in\overline{\mathbb{N}}}\lambda_n$ then π extends to a homomorphism of C^*-algebras from $C^*_\pi(\Gamma_\infty)$ to $C^*_r(G)$.

The hypothesis implies that the trivial representation of Γ_∞ is an isolated point in the dual of $C^*_\pi(\Gamma_\infty)$, and therefore that there is a projection $p \in C^*_\pi(\Gamma_\infty)$ which, for any representation of Γ_∞ weakly contained in the direct sum $\oplus_{n\in\overline{\mathbb{N}}}\lambda_n$, acts as the orthogonal projection onto the Γ_∞-fixed vectors. Let us consider the image $\pi(p)$ of the projection p in the C^*-algebra $C^*_r(G)$. Since the regular representation of Γ_∞ has no fixed vectors it follows that $\pi(p)$ maps to zero in $C^*_r(G_\infty)$. We shall now show that the K-theory class $[\pi(p)] \in K_0(C^*_r(G))$ is not in the image of the K-theory of $C^*_r(G_{\mathbb{N}})$. This will prove the proposition.

Obviously the groupoid $G_{\mathbb{N}}$ is the disjoint union $\coprod_{n\in\mathbb{N}}\Gamma_n$, with the discrete topology. We have $C^*_r(G_{\mathbb{N}}) = \oplus_{n\in\mathbb{N}}C^*_r(\Gamma_n)$. Now it follows from the definition of the norm in $C^*_r(G)$ that the representations $\lambda_n\colon C_c(G) \to C(\Gamma_n)$ extend to $*$-homomorphisms from $C^*_r(G)$ to $C^*_r(\Gamma_n)$. This gives us what we want: the class $[\pi(p)]$ does not lie in the image of $\oplus_{n\in\mathbb{N}}K_0(C^*_r(\Gamma_n))$ because $\lambda_n([\pi(p)]) \in K_0(C^*_r(\Gamma_n))$ is non-zero for every n (since $\lambda_n(\pi(p))$ is non-zero and $C^*_r(\Gamma_n)$ is finite dimensional), while $\lambda_n(x)$ is zero for almost all n whenever x belongs to the image of $\oplus_{n\in\mathbb{N}}K_0(C^*_r(\Gamma_n))$. □

Here are three concrete examples (and therefore three concrete counterexamples to the surjectivity of the Baum–Connes map).

1st Counterexample. Let Γ_∞ be a discrete group with Kazhdan's property (T) (see [HV]) and suppose that the Baum–Connes map $K^{\mathrm{top}}_*(\Gamma_\infty) \to K_*(C^*_r(\Gamma_\infty))$ is injective. Then, no matter what the homomorphisms π_n and groups Γ_n are, the hypotheses in Proposition 3 and the first part of Lemma 1 are satisfied. Therefore the Baum–Connes map $K^{\mathrm{top}}_0(G) \to K_0(C^*_r(G))$ fails to be surjective.

This happens in particular when $\Gamma_\infty = SL_3(\mathbb{Z})$, when $\Gamma_n = SL_3(\mathbb{Z}/n\mathbb{Z})$, and when π_n are the obvious surjections. Indeed, $SL_3(\mathbb{Z})$ is perhaps the exemplar of a property (T) group, while it follows from the work of Kasparov [K3,5] that the Baum–Connes map (even with coefficients) is injective for all closed subgroups of any connected Lie group.

Note that in this case the groupoid G is Hausdorff.

2^{nd} Counterexample. Let Γ_∞ be $SL_2(\mathbb{Z})$ and let $\pi_n : \Gamma_\infty \to SL_2(\mathbb{Z}/n\mathbb{Z})$ be the obvious surjections. A well-known theorem of Selberg (see [Lu]) shows that the hypothesis in Proposition 3 is satisfied. In addition the hypothesis in the second part of Lemma 1 is satisfied since $SL_2(\mathbb{Z})$ is known to be K-amenable (see [Cu] and [K4]). Therefore the map $K_0(C^*_{\max}(G)) \to K_0(C^*_r(G))$ fails to be surjective (and hence the Baum–Connes map for G fails to be surjective also). Once again, G is Hausdorff.

3^{rd} Counterexample. Let Γ_∞ be any non-amenable but K-amenable discrete group (for instance let $\Gamma_\infty = \mathbb{F}_2$, [Cu]) and let $\Gamma_n = 1$ for all $n \in \mathbb{N}$. Then the hypothesis in Proposition 3 is obviously satisfied and in addition the hypothesis in the second case of Lemma 1 is satisfied. Therefore the map $K_0(C^*_{\max}(G)) \to K_0(C^*_r(G))$ and also the Baum–Connes map for G fail to be surjective. Note that this groupoid G is not Hausdorff.

3 Injectivity Counterexamples

In this section, we shall build on the third counterexample of the previous section in order to construct a counterexample to the injectivity of the Baum–Connes map. The groupoid we obtain will not be Hausdorff; we shall present a Hausdorff counterexample in section 5.

Let a and b be two distinct points on the circle S^1. Denote by $[b,a]$ one of the closed arcs connecting a and b and denote by $[a,b]$ the other one. We shall write $]a,b[$ for the interior of $[a,b]$. Let G be the quotient of the constant family of groups $\mathbb{F}_2 \times S^1$ over the topological space S^1 by the following equivalence relation : $(g,x) \sim (h,y)$ if and only if either $x = y \in]a,b[$ or $x = y \in [b,a]$ and $g = h$ (we give G the quotient topology). Thus G is a field of groups over S^1 whose fibers over $[b,a]$ are $\mathbb{F}_2$ and whose fibers over $]a,b[$ are the trivial group 1. Of course G is non-Hausdorff.

The reduced C^*-algebra $C^*_r(G)$ is the completion of $C_c(\mathbb{F}_2 \times S^1)$ in the following C^*-seminorm:

$$\|f\| = \max\Big\{ \sup_{x\in[b,a]} \|f_x\|_{C^*_r(\mathbb{F}_2)}\,,\ \sup_{y\in[a,b]} \Big| \sum_{g\in\mathbb{F}_2} f_y(g) \Big| \Big\},$$

where f_x denotes the restriction of $f\colon \mathbb{F}_2 \times S^1 \to \mathbb{C}$ to $\mathbb{F}_2 \cong \mathbb{F}_2 \times \{x\} \subseteq \mathbb{F}_2 \times S^1$.

We therefore have an obvious $*$-monomorphism

$$\pi\colon C^*_r(G) \to C[b,a] \otimes C^*_r(\mathbb{F}_2) \oplus C[a,b]\,.$$

In fact this $*$-monomorphism is a $*$-isomorphism. To see this, recall first that since $\mathbb{F}_2$ is non-amenable there is an element $f \in C^*_{\max}(\mathbb{F}_2)$ such that the image of f under the trivial representation of $\mathbb{F}_2$ is 1 while the image of f in $C_r^*(\mathbb{F}_2)$ is 0. Now, the obvious inclusion $\mathbb{C}[\mathbb{F}_2] \subseteq C_c(\mathbb{F}_2 \times S^1)$ which associates to a function on $\mathbb{F}_2$ the function on $\mathbb{F}_2 \times S^1$ which is constant in the S^1-direction determines a $*$-homomorphism $\rho\colon C^*_{\max}(\mathbb{F}_2) \to C^*_{\max}(G)$. The image of the element f under the threefold composition of ρ, the map from $C^*_{\max}(G)$ to $C_r^*(G)$, and the map π above, is the element $0 \oplus 1 \in C[b,a] \otimes C_r^*(\mathbb{F}_2) \oplus C[a,b]$. This shows that the image of the $*$-homomorphism π contains $0 \oplus 1$. The remainder of the proof of surjectivity is an easy computation, based on the fact that the compositions of π with the coordinate projections to $C[b,a] \otimes C_r^*(\mathbb{F}_2)$ and to $C[a,b]$ are surjective.

Having shown that π is an isomorphism let us now consider, for $i = 0, 1$, the following commuting diagram:

$$\begin{array}{ccccccccc}
0 & \to & K_i(C[a,b]) & \to & K_i(C_r^*(G)) & \to & K_i(C_r^*(\mathbb{F}_2) \otimes C[b,a]) & \to & 0 \\
 & & \uparrow 0 & & \uparrow & & \uparrow \wr & & \\
0 & \to & K_i(C_0\,]a,b[\,) & \to & K_i(C^*_{\max}(G)) & \to & K_i(C^*_{\max}(\mathbb{F}_2) \otimes C[b,a]) & \to & 0 \\
 & & \uparrow \wr & & \uparrow & & & & \\
 & & K_i^{\mathrm{top}}(\,]a,b[\,) & \to & K_i^{\mathrm{top}}(G)\,. & & & &
\end{array}$$

The bottom left of the diagram incorporates two facts about the full assembly map that we have not yet discussed. First, if U is an open saturated subset of the object space of a groupoid G then associated to the inclusion of G_U into G there is a map from $K_*^{\mathrm{top}}(G_U)$ to $K_*^{\mathrm{top}}(G)$. The full assembly map is compatible with this and with the K-theory map associated to the inclusion of $C^*_{\max}(G_U)$ as an ideal in $C^*_{\max}(G)$. This is another instance of the functoriality of the full assembly map that was discussed in section 1. Second, if U is a locally compact space, and if we consider U as a trivial groupoid (with only identity morphisms), then the Baum–Connes map is an isomorphism for U.

LEMMA 4. *The first two lines of this diagram are (split) exact.*

Proof of the lemma. From our computation of $C_r^*(G)$ it is obvious that the top line is split exact. The fact that the second line is split exact comes from the following commutative diagram:

$$\begin{array}{ccc}
K_i(C^*_{\max}(G)) & \longrightarrow & K_i(C^*_{\max}(\mathbb{F}_2) \otimes C([b,a])) \\
 {\scriptstyle\rho}\nwarrow & & \nearrow{\scriptstyle\simeq} \\
 & K_i(C^*_{\max}(\mathbb{F}_2)) &
\end{array}$$

□

The lemma proves that the image of the generator of $K_0(C[a,b])$ in $K_0(C_r^*(G))$ does not come from $K_0(C^*_{\max}(G))$ and that the image $\beta \in K_1^{\mathrm{top}}(G)$ of the Bott generator of $K_1^{\mathrm{top}}(\,]a,b[\,)$ has a non-zero image in $K_1(C^*_{\max}(G))$. Therefore this class β is non-zero but maps to zero in $K_1(C^*_r(G))$. Hence the Baum–Connes map for G is neither injective nor surjective.

4 Foliation Counterexamples

We now show how to slightly modify the groupoid constructed in the previous section so as to obtain a counterexample which is the holonomy groupoid of a foliation.

Let us first recall the 'foliated bundle construction', which is the construction of a certain foliation associated to a group action. Suppose that a discrete group Γ acts smoothly on a smooth manifold B. Suppose also that M is a smooth manifold and that $\widetilde{M}$ is a Galois covering of M with Galois group Γ. Then the diagonal action of Γ on $\widetilde{M} \times B$ is free and proper, and the quotient space

$$V = (\widetilde{M} \times B)/\Gamma$$

is a smooth manifold (which is compact if M and B are compact). It is foliated by the images of the spaces $\widetilde{M} \times \{x\}$, for $x \in B$.

Returning to the Γ-action on B, form the groupoid with objects B, set of arrows $B \times \Gamma$, range and source maps $(x,g) \mapsto x$ and $(x,g) \mapsto g^{-1} \cdot x$, and composition $(x,g)(y,h) = (x,gh)$ (note that these morphisms are composable precisely when $g^{-1} \cdot x = y$). If we write the action as say $\theta\colon \Gamma \to \mathrm{Diff}(B)$ then let us write the associated groupoid as $B \times_\theta \Gamma$.

Recall that an action of a discrete group Γ on a locally compact space X is said to be *almost free* if for every $g \in \Gamma \setminus \{1\}$, the interior of the set $\{\, x \in X : \; g \cdot x = x \,\}$ is empty. If the smooth action of Γ on B above is almost free then the restriction of the holonomy groupoid to a transversal $\{m\} \times B$ in V is the groupoid $B \times_\theta \Gamma$.

In the non-almost free case, the restricted groupoid is the quotient of the groupoid $B \times_\theta \Gamma$ by the following equivalence relation: $(x,g) \sim (x',g')$ if and only if $x = x'$ and $g'g^{-1}$ acts as the identity on a neighborhood of x in B.

We are now going to build a foliation, starting from a certain family of actions of the free group $\mathbb{F}_2$ by homographic transformations on the projective line $\mathbb{P}^1(\mathbb{R})$. For any $t \in \mathbb{R}$ set

$$\alpha_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Lemma 5. *Let α and β be generators of the free group $\mathbb{F}_2$. For all except* *countably many $t \in \mathbb{R}$, the action of $\mathbb{F}_2$ on $\mathbb{P}^1(\mathbb{R})$ for which α and β act via the matrices α_t and β_t in $SL_2(\mathbb{R})$ is almost free.*

Proof. A non-trivial element $g \in PSL_2(\mathbb{R})$ has at most two fixed points in $\mathbb{P}^1(\mathbb{R})$. It is therefore enough to show that for all but countably many $t \in \mathbb{R}$, the group homomorphism $\varphi_t : \mathbb{F}_2 \to SL_2(\mathbb{R})$ determined by α_t and β_t is injective (note that as $\mathbb{F}_2$ has no torsion, the map $\mathbb{F}_2 \to PSL_2(\mathbb{R})$ will then also be injective). It is well known that for *every* $t \geq 2$ this homomorphism is injective. So for any non-trivial word $w \in \mathbb{F}_2$, the associated matrix $\varphi_t(w) \in SL_2(\mathbb{R})$ is polynomial in t and not identically the identity matrix. It follows that for every non-trivial word w the set $\{t \in \mathbb{R}, \varphi_t(w) = 1\}$ is finite. □

Now, as in the previous section, let a and b be distinct points in S^1, and let $[a,b]$ and $[b,a]$ be the closed arcs connecting them. Let $u : S^1 \to \mathbb{R}$ be a smooth function which vanishes on $[a,b]$ and which is constant on no open interval within $[b,a]$. Consider the action θ of $\mathbb{F}_2$ on $B = S^1 \times \mathbb{P}^1(\mathbb{R})$ given by the following formula on the generators of $\mathbb{F}_2$:

$$\alpha\colon (x,z) \mapsto \big(x, \alpha_{u(x)}(z)\big)$$
$$\beta\colon (x,z) \mapsto \big(x, \beta_{u(x)}(z)\big)\,.$$

Notice that this fibers as a family of group actions over S^1. Let M be a smooth compact manifold and let $\widetilde{M}$ be a Galois covering of M with Galois group $\mathbb{F}_2$ (for instance we could let M be a Riemann surface of genus 2 and consider its Galois covering associated to a surjective homomorphism $\pi_1(M) \to \mathbb{F}_2$). Form the associated foliated manifold V, as above. It follows from our remarks about holonomy groupoids that the restriction of the holonomy groupoid for V to a transversal $\{m\} \times B$ is the quotient of the groupoid $B \times_\theta \mathbb{F}_2$ by the following equivalence relation:

$$\big((x,z),g\big) \sim \big((x',z'),g'\big) \quad \Leftrightarrow \quad \begin{array}{c} x = x' \in [b,a],\ z = z' \text{ and } g = g' \\ \text{or} \\ x = x' \in \,]a,b[\text{ and } z = z'. \end{array}$$

The groupoid G is thus a field of groupoids over S^1 whose fiber at $x \in [b,a]$ is $\mathbb{P}^1(\mathbb{R}) \rtimes_{\varphi_{u(x)}} \mathbb{F}_2$ and whose fiber at $x \in \,]a,b[$ is $\mathbb{P}^1(\mathbb{R})$. With the obvious notation, the restricted groupoid $G_{[b,a]}$ is

$$G_{[b,a]} = \big([b,a] \times \mathbb{P}^1(\mathbb{R})\big) \rtimes_\theta \mathbb{F}_2\,,$$

while $G_{]a,b[}$ is the trivial groupoid $]a,b[\, \times \mathbb{P}^1(\mathbb{R})$.

Just as in the last section, we use the non-amenability of $\mathbb{F}_2$ and the natural morphism $C^*_{\max}(\mathbb{F}_2) \to C^*_r(G)$ to compute that

$$C^*_r(G) \cong C^*_r(G_{[b,a]}) \oplus C\big([a,b] \times \mathbb{P}^1(\mathbb{R})\big)\,.$$

With this in hand, let us consider, for $i = 0, 1$, the following commuting diagram:

$$\begin{array}{ccccccccc} 0 & \to & K_i(C([a,b] \times \mathbb{P}^1(\mathbb{R}))) & \to & K_i(C_r^*(G)) & \to & K_i(C_r^*(G_{[b,a]})) & \to & 0 \\ & & \uparrow 0 & & \uparrow & & \uparrow \wr & & \\ 0 & \to & K_i(C(\,]a,b[\, \times \mathbb{P}^1(\mathbb{R}))) & \to & K_i(C_{\max}^*(G)) & \to & K_i(C_{\max}^*(G_{[b,a]})) & \to & 0 \\ & & \uparrow \wr & & \uparrow & & & & \\ & & K_i^{\mathrm{top}}(\,]a,b[\, \times \mathbb{P}^1(\mathbb{R})) & \to & K_i^{\mathrm{top}}(G) & & & & \end{array}$$

Lemma 6. *The first two lines of this diagram are (split) exact.*

The lemma proves that the image of the generator of $K_0(C([a,b]\times\mathbb{P}^1(\mathbb{R})))$ in $K_0(C_r^*(G))$ does not come from $K_0(C_{\max}^*(G))$, and that the image in $K_0^{\mathrm{top}}(G)$ of the generator of $K_0^{\mathrm{top}}(\,]a,b[\,\times\mathbb{P}^1(\mathbb{R}))$ has a non-zero image in $K_0(C_{\max}^*(G))$ but goes to 0 in $K_0(C_r^*(G))$. Therefore the Baum–Connes map $K_0^{\mathrm{top}}(G) \to K_0(C_r^*(G))$ is neither injective nor surjective.

Proof of the lemma.. The first line is split exact by our computation of $C_r^*(G)$. For the second we consider the following commutative diagram:

$$\begin{array}{ccc} K_i(C_{\max}^*(G)) & \longrightarrow & K_i(C_{\max}^*(G_{[b,a]})) \\ \uparrow & \nearrow & \\ K_i(C(S^1 \times \mathbb{P}^1(\mathbb{R})) \rtimes_{\theta,\max} \mathbb{F}_2) & & \end{array}$$

The diagonal map is (split) surjective because of the following general lemma. Therefore the horizontal map is (split) surjective. □

In the lemma below we shall use the notation $\rtimes_\lambda$ for a crossed product associated to an action λ of a group on a C^*-algebra. The lemma works for either the full or the reduced crossed product.

Lemma 7. *Let A be a C^*-algebra and I a two-sided closed ideal in it. Let λ be an action of $\mathbb{F}_2$ on the C^*-algebra A which leaves I invariant, and denote also by λ the induced actions on I and A/I. If λ is homotopic to λ' through C^*-algebra actions which leave I invariant, and if there is a section $A/I \to A$ that is equivariant with respect to λ', then the sequence*

$$0 \to K_*(I \rtimes_\lambda \mathbb{F}_2) \to K_*(A \rtimes_\lambda \mathbb{F}_2) \to K_*\big((A/I) \rtimes_\lambda \mathbb{F}_2\big) \to 0$$

is split-exact.

Proof. This is a consequence of the Baum–Connes conjecture with coefficients for $\mathbb{F}_2$, or equivalently the Pimsner-Voiculescu sequence for crossed products by $\mathbb{F}_2$. These results imply that if C is an $\mathbb{F}_2$-C^*-algebra and $K_i(C) = 0$ for $i = 0, 1$ then $K_i(C \rtimes \mathbb{F}_2) = 0$ for $i = 0, 1$. If Λ is the action on $A[0,1]$ that makes the homotopy, and still denotes the induced action on $I[0,1]$ and $(A/I)[0,1]$, $K_i(A]0,1]) = 0$ and therefore the evaluation $A[0,1] \rtimes_\Lambda \mathbb{F}_2 \to A \rtimes_\lambda \mathbb{F}_2$ is a K-theory isomorphism and the same is true for the other evaluation and for I or A/I instead of A. We conclude by the following diagram, where all squares commute:

$$\begin{array}{ccccc}
K_i(I \rtimes_\lambda \mathbb{F}_2) & \to & K_i(A \rtimes_\lambda \mathbb{F}_2) & \to & K_i((A/I) \rtimes_\lambda \mathbb{F}_2) \\
\uparrow\wr & & \uparrow\wr & & \uparrow\wr \\
K_i(I[0,1] \rtimes_\Lambda \mathbb{F}_2) & \to & K_i(A[0,1] \rtimes_\Lambda \mathbb{F}_2) & \to & K_i((A/I)[0,1] \rtimes_\Lambda \mathbb{F}_2) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
K_i(I \rtimes_{\lambda'} \mathbb{F}_2) & \to & K_i(A \rtimes_{\lambda'} \mathbb{F}_2) & \to & K_i((A/I) \rtimes_{\lambda'} \mathbb{F}_2)
\end{array} \qquad \square$$

REMARK 8. It follows from this example that in the non-Hausdorff case *there is no fundamental cycle* (see [C]) defined on the C^*-algebra of the foliation. Indeed, although our foliation is transversally oriented, the Bott generator of a small transversal maps to 0 in the reduced C^*-algebra of G. Recall that, in the definition of the fundamental cycle one takes a smooth form ω on G, restricts it to G^0 and then integrates it. Now, if the groupoid is not Hausdorff, G^0 is not closed in G and the definition of $C^\infty(G)$ is modified. In particular, the restriction of a 'closed smooth form in G' is no longer a closed form in G^0. It follows that the formula for the 'fundamental cycle' is no longer a cyclic cocycle.

One should be able to bypass this difficulty by working with the 'max' C^*-algebra of G and integrating over G and slightly modifying the fundamental cycle. Note that the fundamental cycle in K-homology (constructed over the 'max' C^*-algebra) as defined in [HilS] makes sense in the non-Hausdorff setting.

5 An Improved Injectivity Counterexample

The purpose of this section is to present a *Hausdorff* groupoid for which the Baum–Connes map fails to be injective.

Let Γ be a discrete group with property (T), let K be a compact Lie group and $\iota : \Gamma \to K$ an injective group homomorphism with dense image. Let $\mathfrak{k}$ be the Lie algebra of K. Then K, and therefore Γ, act on $\mathfrak{k}$ by the adjoint action and we denote by $\mathfrak{k} \rtimes \Gamma$ the associated semi-direct product group.

The total space of the normal bundle of Γ in K is a group isomorphic to $\mathfrak{k} \rtimes \Gamma$. The deformation to the normal cone (see [HilS]) produces from the inclusion of Γ into K a C^∞-groupoid G_0 which is a field of groups over $\mathbb{R}$ whose fiber at 0 is $\mathfrak{k} \rtimes \Gamma$ and whose restriction to $\mathbb{R} \setminus \{0\}$ is the trivial field with fiber K.

Let $u : S^1 \to \mathbb{R}$ be any C^∞-function which vanishes on $F = [b, a]$ and which is nowhere zero on the complementary interval $U =]a, b[$. Let G be the restriction to the graph of u of the C^∞-groupoid $S^1 \times G_0$ (the latter is a field of groups over $S^1 \times \mathbb{R}$). Then G is a C^∞-groupoid. It is a field of groups over S^1 whose fiber over F is $\mathfrak{k} \rtimes \Gamma$ and whose fiber over U is K.

The K-theory of the full groupoid C^*-algebra is easy to compute via the short exact sequence

$$0 \to C^*_{\max}(G_U) \to C^*_{\max}(G) \to C^*_{\max}(G_F) \to 0\,.$$

Since G restricts to constant fields of groups over U and F we get

$$C^*_{\max}(G_U) \cong C^*(K) \otimes C_0(U) \quad \text{and} \quad C^*_{\max}(G_F) \cong C^*_{\max}(\mathfrak{k} \rtimes \Gamma) \otimes C(F)\,.$$

The trivial representations of the groups K and $\mathfrak{k} \rtimes \Gamma$ yield a homomorphism $\varepsilon : C^*_{\max}(G) \to C(S^1)$.

Let $p_0 \in C^*(K)$ be the projection associated to the trivial representation of K. The product of p_0 with the Bott generator on U determines a K_1-class for $C^*_{\max}(G_U)$ and therefore a K_1-class β for $C^*_{\max}(G)$. Note that $\beta \neq 0$ since its image by ε is the Bott generator of S^1. This class lies in the image of the full assembly map for G, thanks to the commuting diagram

$$\begin{array}{ccc} K_*(C^*_{\max}(G_U)) & \xrightarrow{\cong} & K_*(C^*_{\max}(G)) \\ \uparrow & & \uparrow \\ K^{\text{top}}_*(G_U) & \to & K^{\text{top}}_*(G) \end{array}$$

and the fact that the full assembly map for a constant field of compact groups, such as the groupoid G_U, is an isomorphism.

We are going to show that the class β maps to zero in $K_1(C^*_r(G))$. This will of course prove that the Baum–Connes map for the groupoid G fails to be injective.

Let p be the Kazhdan projection in $C^*_{\max}(\Gamma)$, which acts as the orthogonal projection onto the Γ-fixed vectors in any unitary representation of Γ. In view of the obvious groupoid morphism from $S^1 \times \Gamma$ to G, every element of $C^*_{\max}(\Gamma)$ acts as a multiplier of $C^*_r(G)$ and of $C^*_{\max}(G)$. We can therefore form the C^*-subalgebras $pC^*_{\max}(G)p$ and $pC^*_r(G)p$ of the full and reduced groupoid algebras, respectively. Following the pattern set in earlier sections we find that

$$pC^*_r(G)p \cong C(F)\,.$$

Hence $K_1(pC^*_r(G)p) = 0$. But the class β is contained within the image of the map from $K_1(pC^*_{\max}(G)p)$ to $K_1(C^*_{\max}(G))$, and it therefore follows immediately from the commuting diagram

$$\begin{array}{ccc} K_1(pC^*_r(G)p) & \to & K_1(C^*_r(G)) \\ \uparrow & & \uparrow \\ K_1(pC^*_{\max}(G)p) & \to & K_1(C^*_{\max}(G)) \end{array}$$

that β maps to zero in $K_1(C_r^*(G))$, as required.

6 Coarse Counterexamples

The coarse Baum–Connes conjecture proposes a formula for the K-theory of the *coarse C^*-algebra* of a proper metric space X of bounded coarse geometry. We begin by very briefly reviewing some definitions; for more details see [HiR] and [STY].

We shall assume here that X is *discrete* and of *bounded geometry*, the latter condition meaning that for every $R > 0$ there is some N such that no ball of radius R in X contains more than N points. This does not really entail a loss of generality since every proper metric space of bounded coarse geometry is equivalent within coarse geometry to such a discrete space.

If X is discrete and of bounded geometry then denote by $B(X)$ the space of those $X \times X$ matrices $[a_{xx'}]$ which have the following two properties:

- *Uniform boundedness:* $\sup\{\, |a_{xx'}| : x, x' \in X \,\} < \infty$;
- *Finite propagation:* $\sup\{\, d(x, x') : a_{xx'} \neq 0 \,\} < \infty$.

Then $B(X)$ is a $*$-algebra under the usual matrix operations, and it may be represented faithfully as bounded operators on the Hilbert space $\ell^2(X)$. We denote by $B^*(X)$ its norm-completion. This is what is called the *uniform algebra* or the *uniform Roe algebra* of X.

The *coarse algebra* or *coarse Roe algebra* of X is defined in a similar way, but starting now with the space of matrices $[a_{xx'}]$ whose entries $a_{xx'}$ are compact operators on a fixed separable Hilbert space H. As with the matrices in $B(X)$, the new operator matrices are required to be uniformly norm-bounded and of finite propagation. The $*$-algebra obtained in this way is represented faithfully as bounded operators on $\ell^2(X) \otimes H$, and the coarse C^*-algebra $C^*(X)$ is the norm-completion.

The coarse Baum–Connes theory associates to X certain topological groups $K_*^{\mathrm{top}}(X)$, and the coarse Baum–Connes conjecture is that an associated assembly map

$$\mu\colon K_*^{\mathrm{top}}(X) \to K_*(C^*(X))$$

is an isomorphism of abelian groups. The main reason for studying the coarse Baum–Connes conjecture is the following descent principle (which is very closely related to similar statements in controlled topology): if Γ is a geometrically finite group then the coarse Baum–Connes conjecture for the word-length metric space underlying Γ implies Novikov's higher signature conjecture for manifolds with fundamental group Γ.

Although a geometric construction of counterexamples to the coarse Baum–Connes conjecture is possible, having studied groupoids at some length already in this note it will suit us better to recall that the coarse Baum–Connes conjecture can be reformulated as a special case of the Baum–Connes conjecture for groupoids (with coefficients). We recall from [STY] that the groupoid in question is an étale groupoid $G(X)$ whose object space is the Stone-Čech compactification βX of X. If we view points of βX as $\{0,1\}$-valued, finitely additive measures on X, then a morphism from one such measure, μ_1, to another, μ_2, is represented by a bijection $f\colon X_1 \to X_2$, where:

- X_1 and X_2 are subsets of X, with $\mu_1(X_1) = 1$ and $\mu_2(X_2) = 1$,
- $\mu_1(E) = \mu_2(f[E])$, for every $E \subseteq X_1$,
- $\sup\{\, d(x, f(x)) \,:\, x \in X_1 \,\} < \infty$.

Two such bijections are regarded as defining the same morphism if they are equal modulo null sets. The reduced C^*-algebra of $G(X)$ identifies with the uniform C^*-algebra $B^*(X)$; the coarse algebra $C^*(X)$ identifies with the reduced crossed product associated to a natural action of $G(X)$ on the C^*-algebra $A_X = \ell^\infty(X, \mathcal{K}(H))$. It is shown in [STY] that the coarse Baum–Connes conjecture for X is the same thing as the groupoid Baum–Connes conjecture for $G(X)$, with coefficients in A_X.

In what follows we shall present a counterexample to the groupoid conjecture for $G(X)$ with coefficients in A_X. Actually, to keep the notation simple, we shall drop the coefficients and just deal with $G(X)$ alone. It is elementary to restore the coefficients. We write G instead of $G(X)$.

We shall attack the conjecture using the same weapon that we introduced in section 1. For this we need a closed, saturated subset F of the object space βX, and we shall choose for this purpose the Stone-Čech corona – that is, we define F to be the complement of X in βX (and therefore, in the notation of section 1, the open set U is X). The open subgroupoid G_X which is complementary to G_F is easy to describe: its object space is the discrete set X and there is exactly one morphism between any two objects. Its associated groupoid C^*-algebra identifies with the compact operators on $\ell^2(X)$ under the isomorphism which identifies $C^*_r(G)$ with $B^*(X)$. As in section 1, we will construct a projection in $C^*_r(G)$ whose image in $C^*_r(G_F)$ vanishes. To prove the latter we will use the following lemma:

LEMMA 9. *If an étale Hausdorff groupoid $\mathcal{G}$ acts on a C^*-algebra A, then the map $C_c(\mathcal{G}; A) \to C_0(\mathcal{G}; A)$ extends to an injection (functorial in A) from $A \rtimes_r \mathcal{G}$ to $C_0(\mathcal{G}; A)$.*

Proof. By definition, $A \rtimes_r \mathcal{G}$ is the completion of $C_c(\mathcal{G};A)$ with respect to the norm $f \mapsto \sup\{\|f \star g\|_2 : g \in C_c(\mathcal{G}_x;A),\ \|g\|_2 \le 1,\ x \in \mathcal{G}^0\}$, where $\|\ \|_2$ is the norm of the Hilbert A_x-module $\ell^2(\mathcal{G}_x;A)$. Taking g to be a function vanishing outside $x \in \mathcal{G}_x$, we find $\|f\|_{A\rtimes_r\mathcal{G}} \ge \|f\|_\infty$, whence the map $C_c(\mathcal{G};A) \to C_0(\mathcal{G};A)$ extends to a map u (functorial in A), from $A \rtimes_r \mathcal{G}$ to $C_0(\mathcal{G};A)$. For $f \in A \rtimes_r \mathcal{G}$ and $g \in C_c(\mathcal{G}_x;A)$, we find $\lambda_x(f)(g) = u(f) \star g$, whence u is injective. Here $\lambda_x : A \rtimes_r \mathcal{G} \to \mathcal{L}(\ell^2(\mathcal{G}_x;A))$ is the left regular representation. □

We are now ready to discuss the counterexample, which will use the notion of an expander graph. Recall that a finite, undirected, connected graph X is said to be a (k,ε)-*expander* if

- it has valence k or less (that is, no more than k edges emanate from any vertex),
- the lowest non-zero eigenvalue of the Laplace operator Δ for X is at least ε.

Here the Laplace operator Δ is the linear operator acting on $\ell^2(X^0)$, where X^0 is the vertex set of X, which is defined by the quadratic form

$$\langle f, \Delta f\rangle = \sum_{d(x,x')=1} \left|f(x) - f(x')\right|^2.$$

The sum is over all pairs of adjacent vertices. See [Lu] for a comprehensive treatment of expander graph theory.

Now, fix k and ε and suppose that $\{X_n\}_{n=1}^\infty$ is a sequence of (k,ε)-expander graphs such that the number of vertices in X_n converges to infinity as n tends to infinity (specific examples will be discussed in a moment). Let X be a metric space, as follows:

- the underlying set of X is the disjoint union of the vertex sets of the graphs X_n,
- the distance between two vertices lying in a single X_n is the graph distance – the shortest length of a path connecting them,
- the smallest distance between a vertex of X_n and a vertex of any X_m with $m \ne n$ converges to infinity as $n \to \infty$.

Since the graphs X_n have uniformly bounded valence, the metric space X has bounded geometry.

Define the Laplace operator Δ on $\ell^2(X) \cong \oplus\ell^2(X_n^0)$ to be the direct sum of the Laplace operators Δ_n for the graphs X_n. Since the lowest non-zero eigenvalues for these Δ_n are bounded uniformly away from zero it follows that 0 is an isolated point of the spectrum of the bounded operator Δ.

Now the operator Δ belongs to the C^*-algebra $B^*(X)$ (in fact it belongs to the subalgebra $B(X)$) and it therefore follows from spectral theory that the orthogonal projection p onto the kernel of Δ belongs to the C^*-algebra $B^*(X)$ too. This projection is of course easy to describe: its range is the direct sum of the constant functions in each $\ell^2(X_n^0)$. As a matrix it is block-diagonal with respect to the decomposition $\ell^2(X) \cong \oplus \ell^2(X_n^0)$ and the nth block is the $X_n^0 \times X_n^0$ matrix all of whose entries are $|X_n^0|^{-1}$. Hence the entries $p_{xx'}$ converge to zero as $x, x' \to \infty$. It follows from Lemma 9 that p maps to zero in $C_r^*(G_F)$.

However the class of the projection p in the K-theory group $K_0(B^*(X))$ does not lie in the image of the K-theory map associated to the inclusion of $\mathcal{K}(\ell^2(X))$ into $B^*(X)$. Indeed, by the hypothesis on the pairwise distance of the X_n's, it follows that $C^*(X)$ is contained in the sum of $\mathcal{K}(\ell^2(X))$ with the ℓ^∞-product $\prod_n \mathcal{K}(\ell^2(X_n))$. We therefore get amorphism

$$C^*(X)/\mathcal{K}(\ell^2(X)) \to \prod_n \mathcal{K}(\ell^2(X_n)) \Big/ \bigoplus_n \mathcal{K}(\ell^2(X_n)) \,.$$

The sequence ofthe evaluation maps yields a homomorphism

$$K_0\Big(\prod_n \mathcal{K}(\ell^2(X_n))\Big) \to \mathbb{Z}^{\mathbb{N}}.$$

The image of p through this morphism is $(1, 1, \ldots, 1, \ldots)$, from which fact it follows that the image of p in $K_0\big(\prod_n \mathcal{K}(\ell^2(X_n)) \big/ \bigoplus_n \mathcal{K}(\ell^2(X_n))\big)$ is nonzero.

We have arrived at the following conclusion: if G is the groupoid $G(X)$ associated to the discrete metric space obtained from an expanding sequence, as above, and if F is the Stone-Čech corona $\beta X \setminus X$, then the sequence of K-theory groups

$$K_0(C_r^*(G_X)) \to K_0(C_r^*(G)) \to K_0(C_r^*(G_F))$$

fails to be exact at the middle. As a result, it follows from the observations made in section 1 that either the Baum–Connes map for G_F fails to be injective or the Baum–Connes map for G itself fails to be surjective.

By reintroducing the coefficient algebra A_X that we have up to now suppressed we get the following result:

Proposition 10. *Let the metric space X, the groupoid G and the closed saturated set F be as above. If the Baum–Connes map for the groupoid G_F, with coefficients, is injective then the coarse Baum–Connes map for X fails to be surjective.* □

If Γ is a finitely generated group, if $\{\Gamma_n\}_{n=1}^{\infty}$ is a sequence of finite quotients which increase to infinity in size, and if the trivial representation of Γ is isolated in the 'left regular' representation of Γ on $\oplus_n \ell^2(\Gamma_n)$, then the sequence of Cayley graphs of the Γ_n, formed with respect to a fixed finite generating set of Γ, is an expanding sequence of graphs of the sort we require. If it happens that each $g \in \Gamma$ apart from the identity element maps to the identity element in only finitely many of the Γ_n then the groupoid G_F is easy to compute: it is just the crossed product groupoid associated to the action of the group Γ on the Stone-Čech corona of $\cup_n \Gamma_n$. So if Γ is a group for which the injectivity of the Baum–Connes map, with coefficients, is proved, then it must follow that the coarse Baum–Connes map for the space X fails to be surjective. Hence, for example:

PROPOSITION 11. *Let $k \geq 2$ and form the Cayley graphs of the quotients $SL_k(\mathbb{Z}/n\mathbb{Z})$ with respect to a fixed finite generating set of $SL_k(\mathbb{Z})$. Let $X = \cup_n SL_k(\mathbb{Z}/n\mathbb{Z})$ and assign to X a metric which is the Cayley graph metric on each $SL_k(\mathbb{Z}/n\mathbb{Z})$ and for which the separation of $SL_k(\mathbb{Z}/n\mathbb{Z})$ from its complement in X increases to infinity, as n tends to infinity. Then the coarse Baum–Connes assembly map for the space X fails to be surjective.* □

7 Counterexamples for Group Actions

In this final section we shall consider the Baum–Connes conjecture for group actions on spaces, or in other words the Baum–Connes conjecture for groups, with coefficients in commutative C^*-algebras.

In a recent paper [G1], M. Gromov has announced the existence of finitely generated groups which do not uniformly embed into Hilbert space. His argument, which he sketches very briefly, is based on two things: the simple but elegant observation that no expanding sequence of graphs embeds uniformly in Hilbert space; and a far more intricate probabilistic scheme for constructing groups into whose Cayley graphs certain expanding sequences of graphs may be mapped.

What follows below is almost entirely based on Gromov's constructions. Since the details have not yet appeared, let us make precise what we shall need. We shall assume the existence of a finitely generated discrete group Γ, of a sequence of (k,ε) expanding graphs X_n, as in the previous section, and of a sequence of maps $\varphi_n \colon X_n^0 \to \Gamma$ from the vertex sets of these graphs into Γ such that:

- There is a constant K such that $d(\varphi_n(x), \varphi_n(x')) \leq K d(x,x')$ for all n and for all $x, x' \in X_n^0$. Here the distance is computed in X_n^0 using the graph distance, and in Γ using the left translation invariant word-length metric for some set of generators which will be fixed throughout.

- $\lim_{n\to\infty}(\max\{\#\varphi_n^{-1}[x]/\#X_n : x \in \Gamma\}) = 0$.

We shall prove that there exists a compact metrizable space Z and an action of Γ on Z by homeomorphisms such that the Baum–Connes map fails to be an isomorphism for the reduced crossed product of $C(Z)$ by Γ.

Consider first the (non-separable) abelian C^*-algebra $A = \ell^\infty(\mathbb{N}; c_0(\Gamma))$. The C^*-algebra$A \rtimes_r \Gamma$ is faithfully represented in $\ell^2(\Gamma \times \mathbb{N})$.

For $n \in \mathbb{N}$, define a partial isometry $\theta_n : \ell^2(\Gamma) \to \ell^2(X_n)$ by

$$\theta_n(f)(x) = \frac{1}{\sqrt{\#\varphi_n^{-1}[\varphi_n(x)]}} f(\varphi_n(x))$$

for $f \in \ell^2(\Gamma),\ x \in X_n$. The positive operator

$$D_n = \theta_n^*\Delta_n\theta_n + (1 - \theta_n^*\theta_n)\colon \ell^2(\Gamma) \to \ell^2(\Gamma)$$

has a one-dimensional kernel, which is spanned by the ℓ^2-normalized function $y \mapsto \sqrt{\#\varphi_n^{-1}[y]/\#X_n}$, and on the orthogonal complement of its kernel D_n is bounded below by $\min(\varepsilon, 1)$. Let D be the direct sum of the operators D_n on $\ell^2(\Gamma \times \mathbb{N})$. Then D belongs to the algebra $\widetilde{A \rtimes_r \Gamma}$ obtained by adjoining a unit to $A \rtimes_r \Gamma$ (it is in fact in $\widetilde{C_c(\Gamma, A)}$), the point 0 is isolated in its spectrum, and the orthogonal projection onto the kernel of D, which is by spectral theory an element p of $A \rtimes_r \Gamma \subset \ell^\infty(\mathbb{N}; \mathcal{K}(\ell^2(\Gamma)))$, is the sequence p_n of matrices with entries

$$p_{n,y,z} = \frac{\sqrt{\#\varphi_n^{-1}[y]\#\varphi_n^{-1}[z]}}{\#X_n}\,, \quad y, z \in \Gamma\,.$$

It therefore follows from our hypotheses that the supremum of the matrix coefficients $p_{n,y,z}$ converge to zero as $n \to \infty$. We show that:

- The class of p in $K_0(A \rtimes_r \Gamma)$ does not come from $K_0(c_0(\mathbb{N} \times \Gamma) \rtimes_r \Gamma)$.
- The image of p in $(A/c_0(\mathbb{N} \times \Gamma)) \rtimes_r \Gamma$ is zero.

To prove the first assertion, denote by $\pi_n : \ell^\infty(\mathbb{N}, c_0(\Gamma)) \rtimes_r \Gamma \to c_0(\Gamma) \rtimes \Gamma$ the evaluation at n. Using π_n, we get a map $K_0(A \rtimes_r \Gamma) \to K_0(c_0(\Gamma) \rtimes \Gamma) = \mathbb{Z}$. Since $\pi_n(p) = p_n$ is a rank one projection, we find $\pi_n([p]) = 1$. Therefore the K-theory class of p in $K_0(A \rtimes_r \Gamma)$ does not come from $K_0(c_0(\mathbb{N} \times \Gamma) \rtimes_r \Gamma)$ (which is an algebraic sum $\oplus_{n\in\mathbb{N}}\mathbb{Z}$).

As for the second assertion, the above computation tells us that the function from Γ to A associated with p by Lemma 9 takes its values in

$c_0(\mathbb{N} \times \Gamma)$. By Lemma 9, the image of p is 0, since the function from Γ to $A/c_0(\mathbb{N} \times \Gamma)$ associated with the image of p is 0.

This shows that the K-theory sequence

$$K_0(c_0(\mathbb{N} \times \Gamma) \rtimes_r \Gamma) \to K_0(A \rtimes_r \Gamma) \to K_0\big((A/c_0(\mathbb{N} \times \Gamma)) \rtimes_r \Gamma\big)$$

fails to be exact at its middle term, and from this we obtain our counterexample as before.

Having found a commutative but non-separable counterexample to the Baum–Connes conjecture it follows from a direct limit argument that there is a separable Γ-C^*-subalgebra which is also a counterexample. Taking the one-point compactification of the Gelfand dual of such a separable C^*-subalgebra we obtain a compact metrizable Γ-space for which the Baum–Connes conjecture is false.

Remark 12. The above argument just shows that the Baum–Connes map fails to be surjective for some space, or else it fails to be injective for another. In fact Guoliang Yu has pointed out that the former always occurs: the Baum–Connes map for the crossed-product $A \rtimes_r \Gamma$, in the above notation, is never surjective.

On the other hand, the mapping cone trick in section 1 allows one to construct from our counterexample a second-countable, locally compact Hausdorff space W such that

$$K_0(C_0(W) \rtimes_{\max} \Gamma) = 0 \quad \text{and} \quad K_0(C_0(W) \rtimes_r \Gamma) \neq 0\,.$$

For such a space the Baum–Connes map of course fails to be surjective (although it is injective since in this case the topological K-theory is zero). The same is true of the one-point compactification of W.

Remark 13. N. Ozawa has found a counterexample similar to ours, but with a *trivial* action on a *non-commutative* C^*-algebra instead of a non-trivial action of Γ on a commutative C^*-algebra. One can apply the mapping cone trick to Ozawa's example in order to construct a C^*-algebra A with trivial Γ action such that $K_*^{\mathrm{top}}(\Gamma, A) = K_*(A \rtimes_{\max} \Gamma) = K_*\big(A \otimes_{\max} C^*_{\max}(\Gamma)\big) = 0$ but $K_*(A \rtimes_r \Gamma) = K_*\big(A \otimes_{\min} C^*_r(\Gamma)\big) \neq 0$.

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