Some open problems in algebraic geometry and the Langlands program.

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I thank my teachers and colleagues, in particular Jean-Benoît Bost and Alain Genestier.

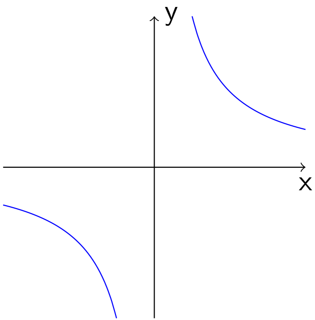
I am very grateful to the CNRS for the permanent position and the great freedom which allowed me to think about big problems.

Freedom is essential to Mathematics. Proofs can be formalized but only humans can appreciate the interest and beauty of mathematical theories, make conjectures based on the harmony of different parts of mathematics, and find promising directions. Moreover we fail most of the time when we try to solve difficult problems, and even when we succeed we often find something different from what we expected.

Algebraic varieties over the complex numbers.

Algebraic varieties are given by algebraic equations.

For example the hyperbola is given by the algebraic equation $xy = 1$. The variety of solutions of this equation when x and y are real numbers is the usual hyperbola.



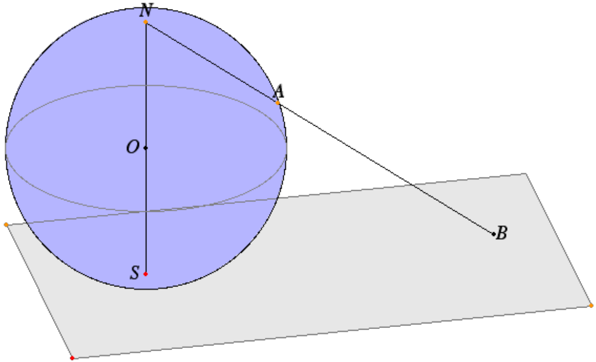
We prefer to consider the variety of solutions of the equation $xy = 1$ when x and y are complex numbers.

We recall that complex numbers are of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

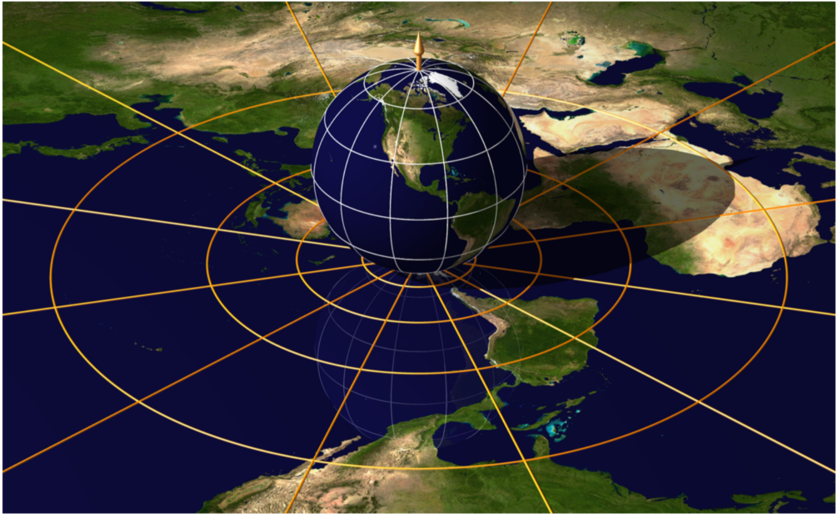
Thus x is now an arbitrary non-zero complex number and $y = 1/x$.
We add two points, where (x, y) is equal to $(0, \infty)$ or $(\infty, 0)$.

In other words x now varies in the plane of complex numbers plus a point at infinity.

Thus the variety of complex solutions of $xy = 1$ together with the two additional points, has the topology of a sphere, which means that if it was made of rubber it could be transformed into a sphere.
Indeed the complex plane plus a point at infinity can be transformed into a sphere by the inverse of the well-known stereographic projection which sends A to B and sends N to the point at infinity on the plane :



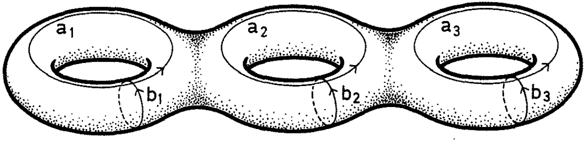
The stereographic projection is used to make maps of earth.



This image is from the beautiful movie « Dimensions » (2008, by Aurélien Alvarez, Étienne Ghys and Jos Leys).

Algebraic varieties Y over the complex numbers have topological invariants which say a lot about them. In particular Poincaré (1854-1912) and others attached to them cohomology groups, denoted by $H^i(Y)$ where i is an indexing integer.

For example the variety Y of complex solutions of the equation $y^2 = x^8 - 1$, to which we add two points at infinity, has the topology of the surface of a donut with three holes



In this example the most interesting cohomology group is $H^1(Y)$. It represents loops up to deformation on this surface. Examples of loops are $a_1, a_2, a_3, b_1, b_2, b_3$.
The other cohomology groups are $H^0(Y)$ and $H^2(Y)$.

In general an algebraic variety Y is given by k variables $x_1, ..., x_k$ and l equations to be satisfied (in the previous example we had two variables x, y and one equation).

The variety Y of complex solutions (together with points at infinity) has an algebraic dimension d (the number of free variables) between $k - l$ and k.

Then its topological dimension is $2d$ (because each complex variable gives topological dimension 2).

The cohomology groups $H^i(Y)$ are defined for $i = 0, 1, ..., 2d$. They indicate how topological subvarieties (generalizing loops in the previous example) can be deformed inside Y.

Rings and fields.

We recall that a ring is equipped with operations $+, \times$. For example we have the ring $\mathbb{Z}$ of relative integers $..., -2, -1, 0, 1, 2, 3, ...$

A ring is called a field if for every $x \neq 0$, $1/x$ exists.

For example we have the field of rational numbers

$$\mathbb{Q} = \{a/b, \quad \text{with} \quad a, b \text{ relative integers }, b \neq 0\}.$$

An example of a rational number is $\frac{2}{5}$.

Ancient Greek geometers proved that $\sqrt{2}$ is not a rational number.

Another field is $\mathbb{Q}(\sqrt{2})$. It consists of all $a + b\sqrt{2}$, with a, b in $\mathbb{Q}$. Multiplication is given by the rule that $\sqrt{2} \times \sqrt{2} = 2$.

The field $\mathbb{Q}(\sqrt{2})$ is equipped with the symmetry

$$\tau : a + b\sqrt{2} \mapsto a - b\sqrt{2}$$

which respects both addition and multiplication.

Symmetries can be composed, for example the composition of τ with τ is the identity because

$$a + b\sqrt{2} \xrightarrow{\tau} a - b\sqrt{2} \xrightarrow{\tau} a + b\sqrt{2}.$$

The symmetries of $\mathbb{Q}(\sqrt{2})$ are exactly the identity and τ.

Because symmetries can be composed we say they form a group.
This is the simplest example of a Galois group.

Some fields have a finite number of elements!

For example $\mathbb{F}_2$ denotes the field of integers modulo 2, which means that we impose $2 = 0$, $3 = 1$ and so on (only parity matters).

The field $\mathbb{F}_2$ has 2 elements $0, 1$ ("even" and "odd") and the operations in $\mathbb{F}_2$ are

$$\begin{aligned} &0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0 \\ &0 \times 0 = 0, \quad 0 \times 1 = 1 \times 0 = 0, \quad 1 \times 1 = 1. \end{aligned} \tag{0.1}$$

This field is much used in computers.

Algebraic varieties over general fields.

We can consider algebraic varieties defined over any field F.

Indeed the operations $+, \times$ are enough to consider algebraic equations with coefficients in F.

The "hyperbola" $xy = 1$ over $\mathbb{F}_2$ is an algebraic variety over $\mathbb{F}_2$.

Its points over $\mathbb{F}_2$ are in finite number, in fact there is just the point $(1, 1)$ (plus the two points at infinity).

It seems that a variety over a finite field like $\mathbb{F}_2$ has nothing to do with topology !

Remarkably, after suggestions of Weil and Serre, Grothendieck defined in the 60s, for any algebraic variety Y over a field F, cohomology groups $H^i(Y)$ thanks to a broad generalization of topology called topos theory.

Moreover these cohomology groups $H^i(Y)$ are equipped with a lot of symmetries (in particular actions of Galois groups).

And when F is a finite field like the field $\mathbb{F}_2$ of integers modulo 2 these cohomology groups $H^i(Y)$ give formulas to count the number of points of Y over F!

It seems that we have plenty of algebraic varieties : we can take as many variables and as many equations as we want !

But the problem often goes in the other direction : we look for an algebraic variety Y such that one of its cohomology groups $H^i(Y)$ (equipped with its symmetries) has some properties ... and we are not able to find Y, although we conjecture it exists.

The cohomology groups $H^i(Y)$ can be cut into parts for reasons coming from algebraic geometry. Such parts of the cohomology groups $H^i(Y)$ come from mysterious objects, which Grothendieck called motives.

The Langlands program.

The Langlands program, first proposed in 1967, involves "global fields", which are of two types : number fields and function fields.

Examples of number fields are $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$.

We will give an example of a function field in the next slide.

Most mathematical problems over number fields have analogues over function fields which are easier. Considering function fields allows to incorporate geometric ideas which are very enlightening.

An example of a function field.

We recall that $\mathbb{F}_2$ is the field of integers modulo 2.

We denote by $\mathbb{F}_2[t]$ the ring of polynomials with coefficients in $\mathbb{F}_2$.

For example $t^3 + t + 1$ is in $\mathbb{F}_2[t]$.
If we write this element in the form 1011 we see that addition and multiplications in $\mathbb{F}_2[t]$ are exactly the same as for integers written in binary representation, except that there is no carry over !

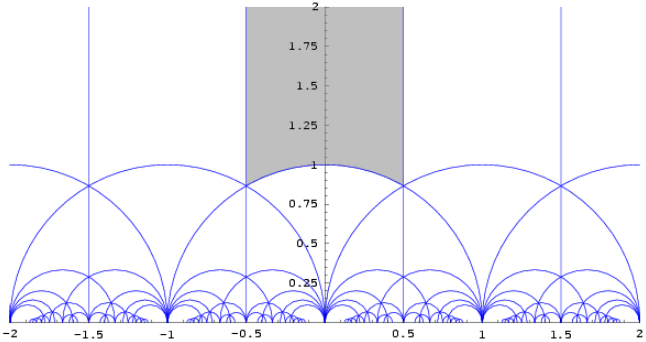
So we see that $\mathbb{F}_2[t]$ is similar to the ring of integers $\mathbb{Z}$ but, because there is no carry over, it is much simpler.

The function field $\mathbb{F}_2(t)$ consists of all quotients P/Q (with $Q \neq 0$) where P and Q are in $\mathbb{F}_2[t]$.

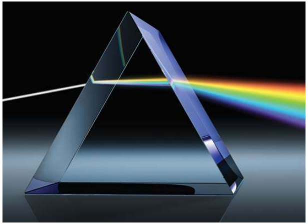
For example $\frac{t+1}{t^3+t+1}$ belongs to the function field $\mathbb{F}_2(t)$.

Automorphic forms.

The Langlands program is mainly concerned with automorphic forms, which are objects of analytic nature, related to both number theory and quantum mechanics/physics. Examples of automorphic forms are functions on the half-plane of complex numbers $a + bi$ with $b > 0$, satisfying some transformation rules such that they are determined by their restriction to any domain in the following tesselation :



The Langlands conjectures predict that automorphic forms for a global field F admit a spectral decomposition, in the same way as light is decomposed into rays of different colors by a prism.



Moreover each "color" of this decomposition (satisfying a condition in the case of number fields) should have an arithmetico-geometric interpretation ; namely, it should be a motive, i.e. a "part of the cohomology" of an algebraic variety Y over F.

This is far from being proven in general :

- in the case of number fields, Shimura varieties give the answer, but only under strong additional conditions,
- in the case of function fields, varieties of Drinfeld shtukas (introduced in the 70s) give the answer in general, at the level of cohomology, but unfortunately not yet in a motivic way.

Three big problems.

1) We want also to understand multiplicities in the spectral decomposition conjectured by the Langlands program. Major works were done by Arthur and Ngo Bao Chau. In the easier case of function fields, Alain Genestier and I have a project about this question.

2) Another conjecture of Langlands, called the “functoriality conjecture”, is very important. I like to see it as a quantization problem. Quantization is the way we pass from classical mechanics to quantum mechanics, it is always a mystery.

3) The Riemann hypothesis (together with its modern generalizations which involve automorphic forms) is the biggest problem in mathematics. The analogues of the Riemann hypothesis for function fields were proven by Deligne in the 70s. Jean-Benoît Bost and I found a relation between this theorem of Deligne and a famous conjecture of Grothendieck.

The ecological crisis.

In the rest of my talk I will turn to something very different from the eternal beauty of arithmetics.
Unfortunately the beauty of our planet is in great danger.

Climate change and the collapse of wild life and biodiversity threaten our future in an unprecedented way.

I asked myself what I can do, as a scientist. I want now to share my ideas. Although they are not original at all, I add the

Disclaimer : all ideas below are only my personal ideas.

I first read what economists say. In fact I enjoyed economics a lot, especially books and reports by Joseph Stiglitz and others, and, independently, utilitarianism (in the sense of John Stuart Mill).

The theory of perfect markets.

Standard economy and the theory of perfect markets recommend in particular

1) greener fiscality to include the huge negative externalities of pollution and destruction of wild life in prices, to give the right incentives to economic agents (for example replace taxes on labor by taxes on pollution),

2) investment in research (which is largely a public good, because it is difficult for private companies in competitive markets like solar panels to invest in research).

Unfortunately, concerning point 2),

- basic research is badly underfunded,
- global public investment in clean energy RD&D is incredibly low : of the order of 0.02% of global GDP (some university grants may not be included in this figure given by the International Energy Agency).

The inertia problem.

Economists say ecology poses a problem of free-riding versus cooperation.

Humans cooperate a lot : even obvious things like the protection of persons and properties depend on human cooperation.

Why do humans cooperate on many things but almost not at all on ecology ?

There is a problem of inertia.

Humans cooperate thanks to institutions : parliaments, justice, governments, the educational, health, tax and pensions systems, central banks, universities, research institutes, and international organizations like UNO, WTO...

These institutions were built after decades or centuries of effort, and after taking the lessons of past disasters. But they were built at a moment when the ecological crisis was not yet perceived.

How to overcome inertia ?

Time is running out to save the planet. After, it will be too late.

Ecological cooperation should have highest priority among the different forms of human cooperation (including trade agreements, debt recognition and protection of property rights) because any other form of human cooperation will become meaningless if the planet is destroyed.

In particular existing institutions should consider ecology as their highest priority, even if they were not created for that purpose.

A very important example is the World Trade Organization because trade rules have a key role to prevent ecological free-riding.

Change trade rules so that each country has a self-interest to adopt a greener fiscality and protect forests and wild life.

Nordhaus proposal

Recent Nobel prize winner William Nordhaus advocates that a club of countries could adopt common ecological taxes , and penalty tariffs against free-riding countries refusing to join the club. Once the club is created, self-interest of countries would be enough to maintain it.

Nordhaus computations suggest that a 5% tariff on imports from countries with carbon price $< 50\$/tCO_2$ would be enough to push almost all countries to adopt carbon price $\geq 50\$/tCO_2$ and global CO_2 emissions would be reduced by a third.

Protection of forests, wild life and biodiversity should absolutely be combined with carbon pricing and be a substitute for it in the case of poor countries or poor areas.

Other ideas are in the Stern-Stiglitz report on carbon pricing, and in many other works.

The role of scientists.

Obvious suggestions :

- invest more in basic research and in R&D for clean energy (including nuclear energy, e.g. thorium molten salt reactors),
- foster and help collaborations between mathematicians, physicists, chemists, biologists on subjects useful for ecology : multidisciplinary institutes could be dedicated to this purpose,
- to have more scientists, help gifted and motivated children develop their abilities (it does not harm other children),
- encourage researchers to turn to subjects useful for ecology (many researchers in CNRS already work on such subjects).

My personal suggestion (to which I would like to contribute) : create a website where physicists, chemists, biologists working on subjects useful for ecology could explain the mathematical problems they have. Interested mathematicians could then browse or search the website and contact them.

Maths for ecology.

Many different mathematical subjects are useful for ecology, for example PDEs, dynamical systems, numerical methods, graph theory, big data, probability, statistics, optimization, optimal control, optimal transport, game theory, mathematical physics, complex problems, computer sciences... I can refer to

- the website "Mathematics for Planet Earth" http ://mpe.dimacs.rutgers.edu
- the book "Mathematics for Planet Earth" edited by Hans Kaper and Christiane Rousseau,
- MathsInTerre, http ://mathsmonde.math.cnrs.fr with a report by Didier Bresch.

As for myself, since I have a background in operator algebras (my first subject), I am now interested in quantum mechanics for new materials for clean energy.