Global Langlands parameterization and shtukas for reductive groups

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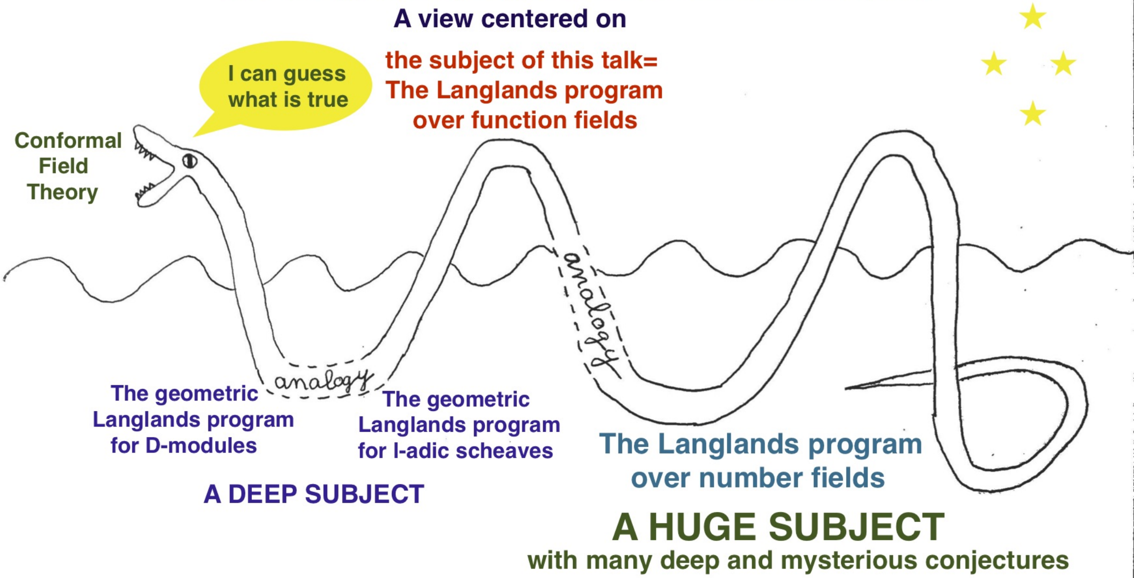
The Langlands program seen as a sea serpent

A view centered on



Preliminaries.

The Langlands program involves arbitrary reductive groups over "global fields", i.e. number fields and function fields.

In this talk a ring means a commutative unital ring, i.e. a set with commutative operations $+, \times$, elements 0, 1 satisfying usual axioms.
An example is the ring $\mathbb{Z}$ of integers.
A field is a ring where any non zero element is invertible.
An example is the field of rational numbers $\mathbb{Q} = \{a/b, a, b \in \mathbb{Z}, b \neq 0\}$.

A number field is a finite extension of $\mathbb{Q}$, i.e. a field generated over $\mathbb{Q}$ by roots of a polynomial with coefficients in $\mathbb{Q}$.

The next 6 slides give a short introduction in the more familiar case of GL_n over $\mathbb{Q}$.

Then the rest of the talk will be about arbitrary reductive groups over function fields.

The case of GL_n over $\mathbb{Q}$: automorphic forms without level.

The vector space of automorphic forms without level for GL_n over $\mathbb{Q}$ is

$$L^2(\mathcal{L}, \mathbb{C}) \quad \text{where} \quad \mathcal{L} = GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R}).$$

The space $\mathcal{L}$ classifies the lattices in $\mathbb{R}^n$, i.e. the couples $(\mathcal{M}, \iota)$, where

- $\mathcal{M}$ is free $\mathbb{Z}$-module of rank n,
- $\iota : \mathcal{M} \to \mathbb{R}^n$ is a discrete embedding of $\mathcal{M}$ in $\mathbb{R}^n$.

Indeed if we fix a basis of $\mathcal{M}$ as a $\mathbb{Z}$-module, i.e. an isomorphism $\mathcal{M} = \mathbb{Z}^n$, ι is given by a matrix in $GL_n(\mathbb{R})$ and to forget the choice of the basis of $\mathcal{M}$ we take the quotient by $GL_n(\mathbb{Z})$.
The Hilbert space $L^2(\mathcal{L}, \mathbb{C})$ is equipped with a unitary representation of $GL_n(\mathbb{R})$ by right translations on $\mathcal{L} = GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})$.

The case of GL_n over $\mathbb{Q}$: unramified Hecke operators.

For any prime number p and any $\lambda = (\lambda_1, ..., \lambda_n)$, where $\lambda_1, ..., \lambda_n$ are integers and $\lambda_1 \geq ... \geq \lambda_n \geq 0$, we have the unramified Hecke operator

$$\begin{aligned} T_{\lambda,p} : L^2(\mathcal{L}, \mathbb{C}) &\to L^2(\mathcal{L}, \mathbb{C}) \\ f &\mapsto \left[(\mathcal{M}, \iota) \mapsto \sum_{\mathcal{M}'} f(\mathcal{M}', \iota|_{\mathcal{M}'})\right] \end{aligned}$$

where the finite sum is taken over all sub-$\mathbb{Z}$-modules $\mathcal{M}' \subset \mathcal{M}$ such that this inclusion is a modification at p with elementary divisors $(\lambda_1, ..., \lambda_n)$, and $\iota|_{\mathcal{M}'}$ is the embedding of $\mathcal{M}'$ in $\mathbb{R}^n$ obtained by restriction of ι to $\mathcal{M}'$.
We say that an inclusion $\mathcal{M}' \subset \mathcal{M}$ is a modification at p with elementary divisors $(\lambda_1, ..., \lambda_n)$ if there exists a basis $(e_1, ..., e_n)$ of $\mathcal{M}$ over $\mathbb{Z}$ such that

$$\mathcal{M}' = \mathbb{Z}p^{\lambda_1}e_1 \bigoplus ... \bigoplus \mathbb{Z}p^{\lambda_n}e_n \quad \text{inside} \quad \mathcal{M} = \mathbb{Z}e_1 \bigoplus ... \bigoplus \mathbb{Z}e_n.$$

For example when $\lambda = (1, 0, ..., 0)$ this is equivalent to $\mathcal{M}/\mathcal{M}' = \mathbb{Z}/p\mathbb{Z}$.

The case of GL_n over $\mathbb{Q}$: automorphic forms with level.

More generally let N be a positive integer. The vector space of automorphic forms with level N is defined by

$$L^2(\mathcal{L}_N, \mathbb{C})$$

where $\mathcal{L}_N$ classifies lattices in $\mathbb{R}^n$ with a level structure, i.e. triples $(\mathcal{M}, \iota, \alpha)$ where $(\mathcal{M}, \iota) \in \mathcal{L}$ and $\alpha : \mathcal{M}/N\mathcal{M} \to (\mathbb{Z}/N\mathbb{Z})^n$ is an isomorphism.
The Hilbert space $L^2(\mathcal{L}_N, \mathbb{C})$ is equipped with the action of

- unramified Hecke operators $T_{\lambda,p}$ as before, but only for p prime to N,
- a non commutative algebra of ramified Hecke operators (not defined in this talk) for each prime number p dividing N
- $GL_n(\mathbb{R})$ by right translation.

Remark. The well-known holomorphic modular forms are obtained, for $n = 2$, as the isotypical parts of $L^2(\mathcal{L}_N, \mathbb{C})$ corresponding to the discrete series of representations of $GL_2(\mathbb{R})$. For $n = 1$ we obtain the Dirichlet characters.

The case of GL_n over $\mathbb{Q}$: the goal of the Langlands program.

The unramified Hecke operators $T_{\lambda,p}$ (for all p prime to N and for all λ) commute with each other and we can simultaneously diagonalize them.
In the setting of the previous slide, the goal of the Langlands program is

- to decompose $L^2(\mathcal{L}_N, \mathbb{C})$ as a direct sum (or rather an integral) of eigenspaces for the $T_{\lambda,p}$, indexed by global Langlands parameters,
- to have a multiplicity formula for these eigenspaces, as representations of the ramified Hecke algebras and of $GL_n(\mathbb{R})$ (in fact these multiplicities are well known in the case of GL_n, but not for general reductive groups and formulas were conjectured by Arthur in general).

The global Langlands parameters are representations of rank n of the Galois group of $\mathbb{Q}$, under some algebraicity conditions on the representations of $GL_n(\mathbb{R})$ which appear (these conditions have no analogue over function fields and we do not explain them for this reason).

The case of GL_n over $\mathbb{Q}$: places of $\mathbb{Q}$.

In general a place of a global field F is a norm (up to equivalence, with a canonical choice in the equivalence class), and the completion of F for such a norm is called a "local field".
The places of $\mathbb{Q}$ are

- the archimedean place, where the completion is $\mathbb{R}$ and the archimedean norm is the usual absolute value,
- for every prime number p, the place p where the completion is $\mathbb{Q}_p$ and the p-adic norm of a number $r \in \mathbb{Q}^*$ is $p^{-\alpha}$, where $r = p^\alpha a/b$ with a, b integers non divisible by p, so that the bigger the power of p dividing r the smaller the p-adic norm of r.

For any p, we recall that $\mathbb{Z}_p$ is the ring of elements of $\mathbb{Q}_p$ of norm ≤ 1. It is also the completion of $\mathbb{Z}$ for the p-adic norm, and also the projective limit $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.
We have the product formula : for any element of $\mathbb{Q}^*$ the product of the archimedean norm and all p-adic norms is 1. For example, $|6|_{\mathbb{R}} = 6$, $|6|_{\mathbb{Q}_2} = 1/2$, $|6|_{\mathbb{Q}_3} = 1/3$ and other norms are 1.

The case of GL_n over $\mathbb{Q}$: adeles.

We define the restricted product $\prod'_p \mathbb{Q}_p$ as the subring of elements of $\prod_p \mathbb{Q}_p$ which belong to $\mathbb{Z}_p$ for all p but finitely many.
We define the ring $\mathbb{A}$ of adèles of $\mathbb{Q}$ as the ring $\prod'_p \mathbb{Q}_p \times \mathbb{R}$.
Then $\mathbb{A}$ is a locally compact ring containing $\mathbb{Q}$ discretely. Indeed, by the product formula, an element of $\mathbb{Q}^*$ cannot be small in $\mathbb{A}$. Moreover $\mathbb{A}/\mathbb{Q}$ is compact.

An element of $\mathbb{Q}$ is in $\mathbb{Z}$ if and only if it has no denominator in p for any prime number p, i.e. if its image in $\prod'_p \mathbb{Q}_p$ is in $\prod_p \mathbb{Z}_p$.
Therefore $GL_n(\mathbb{Z}) \subset GL_n(\mathbb{Q})$ is the inverse image of $GL_n(\prod_p \mathbb{Z}_p) \subset GL_n(\prod'_p \mathbb{Q}_p)$.

We can deduce that $\mathcal{L} = GL_n(\mathbb{Z})\backslash GL_n(\mathbb{R})$ is equal to $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})/GL_n(\prod_p \mathbb{Z}_p)$.
More generally, for any level N,

$$\mathcal{L}_N = GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})/K_N$$

where $K_N = \mathrm{Ker}(GL_n(\prod_p \mathbb{Z}_p) \to GL_n(\mathbb{Z}/N\mathbb{Z}))$ is an open compact subgroup of finite index in $GL_n(\prod_p \mathbb{Z}_p)$.

The general setting for the Langlands program with arbitrary reductive groups.

Let G be a reductive group over a global field F. We assume G is split to simplify.
We denote by $\widehat{G}$ the Langlands dual group of G. It is the split reductive group whose weights and roots are the coweights and coroots of G. Examples :

G	$\widehat{G}$
GL_n	GL_n
SL_n	PGL_n
SO_{2n+1}	Sp_{2n}
Sp_{2n}	SO_{2n+1}
SO_{2n}	SO_{2n}

and if G is one of the five exceptional groups, $\widehat{G}$ is of the same type.
The locally compact ring of adèles $\mathbb{A}$ of F contains F discretely, and the goal of the Langlands program is to decompose $L^2(G(F)\backslash G(\mathbb{A}), \mathbb{C})$, as a representation of $G(\mathbb{A})$, in terms of global Langlands parameters, which are (under some algebraicity conditions in the case of number fields) morphisms from the Galois group of F to $\widehat{G}$.
From now on we consider only function fields.

Some basic notions of algebraic geometry.

For any ring A, Grothendieck defined the "affine scheme" $\operatorname{Spec}(A)$, such that the ring of functions on $\operatorname{Spec}(A)$ is A. For any ideal $I \subset A$, $\operatorname{Spec}(A/I)$ is a closed subscheme of $\operatorname{Spec}(A)$, such that the restriction of functions from $\operatorname{Spec}(A)$ to $\operatorname{Spec}(A/I)$ is the quotient morphism $A \to A/I$. He defined general schemes by gluing affine schemes along open subschemes.

For example, if N is a positive integer, $\operatorname{Spec}(\mathbb{Z}/N\mathbb{Z})$ is a closed subscheme of $\operatorname{Spec}(\mathbb{Z})$. This gives an alternative definition of the level as a subscheme, which will be useful in the case of function fields.

When k is a field and $A \supset k$ (we say that A is a k-algebra), $\operatorname{Spec}(A)$ is called an affine scheme over k. If in addition A has no non zero nilpotent elements, $\operatorname{Spec}(A)$ is called an affine algebraic variety over k. For example $\mathbb{A}^n = \operatorname{Spec} k[t_1, ..., t_n]$ is called the affine space of dimension n.

When a field is finite, its cardinal q is always a power of a prime number, and it is denoted by $\mathbb{F}_q$. If q is a prime number, $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$.

For any $\mathbb{F}_q$-algebra A, the Frobenius morphism $A \to A, x \mapsto x^q$ is an endomorphism of $\mathbb{F}_q$-algebras. For any scheme S over $\mathbb{F}_q$ we denote by $\operatorname{Frob}_S : S \to S$ the morphism acting on functions by $\operatorname{Frob}_S^*(f) = f^q$.

Definition of function fields.

A function field F is the field of rational functions on a smooth projective curve X over a finite field $\mathbb{F}_q$.
A curve means an algebraic variety of dimension 1, smooth means nonsingular and projective means that we add the points at infinity.
A rational function is an algebraic function with arbitrary poles.

The simplest example is $F = \mathbb{F}_q(t)$, the field consisting of all P/Q, with P, Q in the ring $\mathbb{F}_q[t]$ of polynomials in t and $Q \neq 0$. Denoting by t the coordinate on the affine line $\mathbb{A}^1$ we see that $\mathbb{F}_q[t]$ is the ring of functions on $\mathbb{A}^1$ and $F = \mathbb{F}_q(t)$ is the field of rational functions on the projective line $X = \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$.
We can consider $\mathbb{F}_q(t)$ as an analogue of $\mathbb{Q}$ and $\mathbb{F}_q[t]$ as an analogue of $\mathbb{Z}$. The order of the pole (or minus the order of the zero) at ∞ for non zero elements of $\mathbb{F}_q(t)$ is an analogue of $\log |.|_{\mathbb{R}}$ for non zero elements of $\mathbb{Q}$. We note that $\mathbb{F}_q[t]$ is a unique factorization ring like $\mathbb{Z}$ and that unitary irreducible polynomials in $\mathbb{F}_q[t]$ play the same role as prime numbers in $\mathbb{Z}$.

Places of F=closed points of the curve X.

We recall that X is a smooth projective curve over a finite field $\mathbb{F}_q$ and F is its field of rational functions.
We define the closed points of X as the irreducible subschemes of X of dimension 0.
For any closed point v the ring of functions on v is a finite extension of $\mathbb{F}_q$ denoted by $k(v)$ and called the residue field at v.
For example, when $X = \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, the closed points are

- ∞ (where the residue field $k(\infty)$ is $\mathbb{F}_q$),
- for any unitary irreducible polynomial P in $\mathbb{F}_q[t]$, we have a closed point $v \subset \mathbb{A}^1$ such that $k(v) = \mathbb{F}_q[t]/P$ is a finite extension of $\mathbb{F}_q$ of degree $\deg(P)$ and the quotient morphism $\mathbb{F}_q[t] \to \mathbb{F}_q[t]/P$ is the restriction from functions on $\mathbb{A}^1$ to functions on v.

The closed points of X are exactly the places of F : for each closed point v, we denote by F_v the completion of F for the norm $|.|_v$ such that for $f \in F^*$, $|f|_v = (\sharp k(v))^{-\operatorname{ord}_v(f)}$ where $\operatorname{ord}_v(f)$ is the order of vanishing (also called order of the zero) of f at v. Its ring of integers $\mathcal{O}_{F_v} = \{a \in F_v, |a|_v \leq 1\}$ is the ring of functions on the formal neighborhood of v in X.

Geometric interpretation of the adelic quotient.

We recall that the ring of adèles of F is the restricted product $\prod'_v F_v$, consisting of elements of the product which belong to $\mathcal{O}_{F_v}$ for all v but finitely many.
We denote by $\mathbb{O} = \prod_v \mathcal{O}_{F_v}$ the ring of integral adèles.
We recall that G denotes a split reductive group over F.
We have

$$G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) = \mathrm{Bun}_G(\mathbb{F}_q) \tag{0.1}$$

where $\mathrm{Bun}_G(\mathbb{F}_q)$ is the set of isomorphism classes of G-principal bundles over X.
We recall that a G-principal bundle over X is defined as a morphism $Y \to X$ equipped with a simply transitive action of G on the fibers. The GL_r-principal bundles can be equivalently seen as the frame bundles of the vector bundles of rank r.
Equality (0.1) holds because any G-principal bundle over X can be trivialized over $X \setminus S$ where S is a finite set of places of X, and is then given by an element of $\prod_{v \in S} G(F_v)/G(\mathcal{O}_{F_v})$. Moreover $G(\mathbb{A})/G(\mathbb{O})$ is the union of all $\prod_{v \in S} G(F_v)/G(\mathcal{O}_{F_v})$ where S varies, and two trivializations of a G-principal bundle over $X \setminus S$ for some S are related by the action of an element of $G(F)$.

Definition of automorphic forms over function fields.

Let N be a level, i.e. a finite subscheme of X (which is the same as a finite subset of places of X with multiplicities).
Let $\mathcal{O}_N$ be the ring of functions on N. We note that $G(\mathcal{O}_N)$ is a finite group. We define $K_N = \mathrm{Ker}(G(\mathbb{O}) \to G(\mathcal{O}_N))$. It is an open compact subgroup of $G(\mathbb{A})$.
Then we have

$$G(F)\backslash G(\mathbb{A})/K_N = \mathrm{Bun}_{G,N}(\mathbb{F}_q)$$

where $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$ is the set of isomorphism classes of G-principal bundles over X together with a trivialization of their restriction to N.

Definition. An automorphic form with level N is a function on $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$.
In particular an automorphic form with trivial level is a function on $G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) = \mathrm{Bun}_G(\mathbb{F}_q)$.

Stacks.

In fact, as G-principal bundles over X may have automorphisms, $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$ is a groupoid whose elements have finite automorphism groups.
It is equal to the groupoid of the points over $\mathbb{F}_q$ of the stack $\mathrm{Bun}_{G,N}$ over $\mathbb{F}_q$ whose "points" over a scheme S over $\mathbb{F}_q$ (by which we mean morphisms $S \to \mathrm{Bun}_{G,N}$) classify the G-principal bundles over $X \times S$ together with a trivialization of their restriction to $N \times S$.
The products $X \times S$ and $N \times S$ are products of schemes over $\mathrm{Spec}(\mathbb{F}_q)$. To explain what it means in the case of affine schemes, we say that if A and B are $\mathbb{F}_q$-algebras, $\mathrm{Spec}(A) \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_{\mathbb{F}_q} B)$.

A stack is like an algebraic variety whose points may have algebraic automorphism groups. For example the quotient of an algebraic variety by the action of an algebraic group is a stack.

Definition of the vector space of cuspidal automorphic forms over function fields.

Let ℓ be a prime number not dividing q.
We recall that $\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n\mathbb{Z}$ and $\mathbb{Q}_\ell = \mathbb{Z}_\ell[1/\ell]$. Let $\overline{\mathbb{Q}_\ell}$ be an algebraic closure of $\mathbb{Q}_\ell$, i.e. it is obtained by adding to $\mathbb{Q}_\ell$ all the roots of all polynomials with coefficients in $\mathbb{Q}_\ell$.
We write

$$\mathfrak{Aut} = C_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$$

the $\overline{\mathbb{Q}_\ell}$-vector space formed by "cuspidal" functions on $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$ (considered as a set). The cuspidal automorphic forms are the "elementary bricks" to build all automorphic forms and it is enough to understand them.
We can define cuspidal automorphic forms with coefficients in $\mathbb{Q}$. We take them with coefficients in $\overline{\mathbb{Q}_\ell}$ because the ℓ-adic cohomology we need to use and the Langlands parameters we want to construct are both with coefficients in $\overline{\mathbb{Q}_\ell}$.

Definition of the unramified Hecke operators.

We assume first that N is empty. Let v be a closed point of X.
If $\mathcal{G}$ and $\mathcal{G}'$ are two G-principal bundles over X we say that $\mathcal{G}'$ is a modification of $\mathcal{G}$ at v if we are given an isomorphism between their restrictions to $X \setminus v$. Then their relative position $[\mathcal{G}' : \mathcal{G}]$ at v is a dominant coweight λ of G (when $G = GL_n$ it is the n-uple of the elementary divisors). We introduce the unramified Hecke operator

$$T_{\lambda,v} : C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) \to C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$$
$$f \mapsto [\mathcal{G} \mapsto \sum_{\mathcal{G}', [\mathcal{G}':\mathcal{G}]=\lambda} f(\mathcal{G}')]$$

where the finite sum is taken over all the modifications $\mathcal{G}'$ of $\mathcal{G}$ at v with relative position λ.
In the same way, with a level N, and for any closed point v in $X \setminus N$, and any coweight λ, we have $T_{\lambda,v}$ acting on $\mathfrak{Aut} = C_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$.
When λ varies the operators $T_{\lambda,v}$ span the unramified Hecke algebra $\mathcal{H}_v$ which is commutative and acts on $\mathfrak{Aut}$.

Definition of the global Langlands parameters.

Let $\overline{F}$ be an algebraic closure of the function field F, i.e. it is obtained by adding to F all the roots of all polynomials with coefficients in F.
We denote by $\mathrm{Gal}(\overline{F}/F)$ the Galois group of automorphisms of $\overline{F}$ which are Id on F.
It is a profinite group.
For any open subscheme $U \subset X$ (the complement of a finite number of closed points), we denote by $\overline{F}^U \subset \overline{F}$ the subfield generated by all finite extensions of F associated to unramified coverings of U. Then $\mathrm{Gal}(\overline{F}/F)$ acts on $\overline{F}^U$ by a quotient denoted by $\pi_1(U)$ (with base point $\mathrm{Spec}\,\overline{F}$), which is an analogue of the Poincaré group in topology.

Definition. A global Langlands parameter is a $\widehat{G}(\overline{\mathbb{Q}_\ell})$-conjugacy class of continuous and semisimple morphisms $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$, factorizing through $\pi_1(U)$ for some open subscheme $U \subset X$.

We use $\overline{\mathbb{Q}_\ell}$ and not $\mathbb{C}$ because the continuous morphisms $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\mathbb{C})$ have finite image and it turns out that there are not enough of them to parameterize all automorphic forms.

Statement of the main theorem.

To simplify the statement of the theorem we assume from now on that G is semisimple, i.e. its center is finite. Then $\mathfrak{Aut}$ is a $\overline{\mathbb{Q}_\ell}$-vector space of finite dimension.

Theorem. We have a canonical decomposition $\mathfrak{Aut} = \bigoplus_\sigma \mathfrak{H}_\sigma$ indexed by global Langlands parameter $\sigma : \pi_1(X \setminus N) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$. It is respected by all Hecke operators, and compatible with the Satake isomorphism at all closed points of $X \setminus N$.

Unfortunately we don't know the multiplicities for the spaces $\mathfrak{H}_\sigma$ as modules over the Hecke algebras, although multiplicity formulas were conjectured by Arthur.
In the case where $G = GL_n$ everything was already known from the works of Drinfeld for $n = 2$ and of Laurent Lafforgue for arbitrary n. Their method used stacks of shtukas and the Arthur-Selberg trace formula.

The proof of the theorem above uses

- more general stacks of shtukas (introduced by Drinfeld and Varshavsky)
- the geometric Satake equivalence (due to Lusztig, Drinfeld, Ginzburg, and Mirkovic–Vilonen).

Meaning of the compatibility with the Satake isomorphism stated in the theorem.

The Satake isomorphism is a canonical isomorphism

$$[V] \mapsto T_{V,v}$$

from the $\overline{\mathbb{Q}_\ell}$-algebra of representations of $\widehat{G}$ to the unramifed Hecke algebra $\mathcal{H}_v$ (if V is an irreducible representation of $\widehat{G}$, $T_{V,v}$ is a combination of the $T_{\lambda,v}$ for λ a weight of V).

We have $\pi_1(v) := \mathrm{Gal}(\overline{k(v)}/k(v)) = \widehat{\mathbb{Z}}$ with generator $\mathrm{Frob}_v : x \mapsto x^{q^d}$ where d is the degree of v (such that the cardinal of $k(v)$ is q^d).

We still denote by $\mathrm{Frob}_v \in \pi_1(X \setminus N)$ the image of $\mathrm{Frob}_v \in \pi_1(v)$ by the morphism $\pi_1(v) \to \pi_1(X \setminus N)$ (to understand this morphism remember that $v \subset X \setminus N$ and, by analogy with topology, that any morphism of schemes $Y \to Z$ gives a morphism of groups $\pi_1(Y) \to \pi_1(Z)$). The element $\mathrm{Frob}_v \in \pi_1(X \setminus N)$ is well defined up to conjugacy, and called a Frobenius element at v.

Then the compatibility of the decomposition $\mathfrak{Aut} = \bigoplus_\sigma \mathfrak{H}_\sigma$ with the Satake isomorphism means that for any closed point v of $X \setminus N$ and for any representation V of $\widehat{G}$, $T_{V,v}$ preserves this decomposition and acts on $\mathfrak{H}_\sigma$ by multiplication by $\mathrm{Tr}_V(\sigma(\mathrm{Frob}_v))$.

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Idea of the proof of the theorem.

In the 15 next slides we explain the idea of the proof of the theorem. To simplify we assume that N is empty.

We will construct a commutative algebra $\mathcal{B}$ of "excursion operators" containing all the $\mathcal{H}_v$ and such that

- $\mathcal{B}$ acts naturally on $\mathfrak{Aut}$
- each character ν of $\mathcal{B}$ corresponds in a unique way to a Langlands parameter σ.

Since $\mathcal{B}$ is commutative we will obtain a canonical spectral decomposition

$$\mathfrak{Aut} = \bigoplus_{\nu} \mathfrak{H}_{\nu}$$

where the sum is taken over the characters ν of $\mathcal{B}$. By associating to every ν a Langlands parameter σ we will deduce the decomposition of the theorem

$$\mathfrak{Aut} = \bigoplus_{\sigma} \mathfrak{H}_{\sigma}.$$

Sites and topos (1).

To construct this algebra $\mathcal{B}$ we will use the ℓ-adic cohomology of the stacks of shtukas. The ℓ-adic cohomology of stacks (of characteristic $\neq \ell$) is very similar to the Betti cohomology of complex varieties, but it has coefficients in $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}_\ell}$. To define it Grothendieck introduced the notion of topos, an extraordinary generalization of the usual notion of topological space.
To a topological space Y we can associate the category whose

- objects are the open subsets $U \subset Y$
- arrows $U \to V$ are the inclusions $U \subset V$

and we have the notion of covering of an open subset by a family of open subsets.

Sites and topos (2).

A site is an abstract category with a notion of covering of objects by families of objects, satisfying some natural axioms. A sheaf of sets $\mathcal{F}$ on a site is a contravariant functor $U \mapsto \mathcal{F}(U) =$"set of sections of $\mathcal{F}$ over U", such that for any covering of U by a family $(U_i)_{i\in I}$, a section of $\mathcal{F}$ over U is the same as a family of sections of $\mathcal{F}$ over U_i satisfying a gluing condition. We can associate to any sheaf of abelian groups on a site its Čech cohomology exactly as on a topological space.

Although it is not necessary for what follows, we mention the notion of topos introduced by Gothendieck. A topos is the category of sheaves of sets on a site. It is the most fundamental notion. For example a Čech cohomology can be associated to any "object in abelian groups" of a topos.

The Etale Site.

To define the étale cohomology of an algebraic variety Y (say smooth to simplify) Grothendieck introduced the étale site whose objects are the étale morphisms

$$\begin{array}{c} U \\ \downarrow \\ Y \end{array}$$

(a morphism is étale if its differential is everywhere inversible), whose arrows are given by commutative triangles of étale morphisms

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

and with the obvious notion of covering.

Etale cohomology.

The étale cohomology of an algebraic variety Y over an algebraically closed field of characteristic $\neq \ell$ is defined as the Čech cohomology of the étale site of Y with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$, whence $\mathbb{Z}_\ell$ by passing to the projective limit, and $\mathbb{Q}_\ell$ by inverting ℓ.

If Y is defined over a general field k of characteristic $\neq \ell$, $H^*(Y_{\overline{k}}, \mathbb{Q}_\ell)$ is equipped with a continuous action of $\mathrm{Gal}(\overline{k}/k)$ (in the same way as, if Y is an algebraic variety over $\mathbb{R}$, the Betti cohomology of $Y_\mathbb{C}$ is equipped with an action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$).

More generally, we can define the étale cohomology of any stack Y over an algebraically closed field, with coefficients in any $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ (which means roughly a sheaf of $\overline{\mathbb{Q}_\ell}$-vector spaces on Y).

Definition of the stack of shtukas.

For any scheme S over $\mathbb{F}_q$ we recall that $\mathrm{Frob}_S : S \to S$ is the morphism acting on functions by $\mathrm{Frob}_S^*(f) = f^q$.

Let I be a finite set. We define Sht_I as the stack over X^I whose "points" over a scheme S over $\mathbb{F}_q$ (by which we mean morphisms $S \to \mathrm{Sht}_I$) classify shtukas, namely

- points $(x_i)_{i\in I} : S \to X^I$, called the legs of the shtuka,
- a G-principal bundle $\mathcal{G}$ over $X \times S$,
- an isomorphism

$$\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$$

  where $\Gamma_{x_i} \subset X \times S$ denotes the graph of x_i.

It is a Deligne-Mumford stack (i.e. the automorphism groups of points are finite).

The stack of shtukas without legs $\mathrm{Sht}_\emptyset$ is equal to the groupoid $\mathrm{Bun}_G(\mathbb{F}_q)$.

Remark. Shtukas do not have analogues over number fields in general because nobody knows what $(\mathrm{Spec}(\mathbb{Z}))^I$ should be for $\sharp I > 1$. Remarkably Scholze defined an analogue of local shtukas over $\mathbb{Q}_p$.

The geometric Satake equivalence.

We define $\mathcal{M}_I$ as the stack over X^I whose points over a scheme S over $\mathbb{F}_q$ classify

- points $(x_i)_{i\in I} : S \to X^I$,
- G-principal bundles $\mathcal{G}$ and $\mathcal{G}'$ over the formal completion $\widehat{X \times S}$ of $X \times S$ along the union of the Γ_{x_i},
- an isomorphism

$$\phi : \mathcal{G}|_{\widehat{X\times S} \smallsetminus (\bigcup_{i\in I} \Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'|_{\widehat{X\times S} \smallsetminus (\bigcup_{i\in I} \Gamma_{x_i})}$$

  where Γ_{x_i} denotes the graph of x_i.

Thus $\mathcal{M}_I(S)$ depends only on $\bigcup_{i\in I} \Gamma_{x_i}$.
Fusion of legs is what happens when some x_i become equal.
The geometric Satake equivalence associates to any finite set I and any finite dimensional $\overline{\mathbb{Q}_\ell}$-linear representation W of $\widehat{G}^I$ a perverse sheaf $\mathcal{S}_{I,W}$ on $\mathcal{M}_I$, which is functorial in W and compatible with the fusion of legs.

The obvious forgetful morphism $\alpha : \mathrm{Sht}_I \to \mathcal{M}_I$ is smooth.
We define a perverse sheaf $\mathcal{F}_{I,W}$ on Sht_I as the pull-back $\alpha^*(\mathcal{S}_{I,W})$.

The ℓ-adic cohomology of the stacks of shtukas.

We denote by $H_{I,W}$ a $\overline{\mathbb{Q}_\ell}$-vector subspace of the ℓ-adic cohomology with compact support of the fiber of Sht_I over a geometric generic point of X^I (or, in fact equivalently, over a geometric generic point of the diagonal $X \subset X^I$) with coefficients in $\mathcal{F}_{I,W}$. This subspace is defined by a technical condition of Hecke-finiteness. By the work of Cong Xue, it may equivalently be defined by a cuspidality condition, and it is of finite dimension over $\overline{\mathbb{Q}_\ell}$.

We note that what matters is not the total space Sht_I but the morphism

$$\begin{array}{c}\mathrm{Sht}_I\\ \downarrow\\ X^I\end{array}$$

(which associates to a shtuka the I-uple of its legs).

The $\overline{\mathbb{Q}_\ell}$-vector space $H_{I,W}$ is equipped with a continuous action of $(\mathrm{Gal}(\overline{F}/F))^I$ (thanks to partial Frobenius morphisms introduced by Drinfeld).

The strategy to construct the algebra $\mathcal{B}$ of excursion operators.

When $I = \emptyset$ and $W = \mathbf{1}$ (the trivial one-dimensional representation), we have an isomorphism

$$H_{\emptyset,\mathbf{1}} \simeq C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \mathfrak{Aut}$$

because $\mathrm{Sht}_\emptyset = \mathrm{Bun}_G(\mathbb{F}_q)$, and $\mathcal{F}_{\emptyset,\mathbf{1}}$ is the constant sheaf $\overline{\mathbb{Q}_\ell}$.
Thus we want to construct a decomposition

$$H_{\emptyset,\mathbf{1}} = \bigoplus_\sigma \mathfrak{H}_\sigma.$$

The idea is to consider $H_{\emptyset,\mathbf{1}}$ as the "vector space of quantum states with zero particles" and $H_{I,W}$ as the "vector space of quantum states with particles indexed by I with spins given by W". Then the operators in $\mathcal{B} \subset \mathrm{End}(H_{\emptyset,\mathbf{1}})$ are obtained by creating particles, making them interact, and annihilating them, as we shall see more precisely later.

Properties of the $H_{I,W}$

a) Functoriality of $H_{I,W}$ in W

For any finite set I,

$$W \mapsto H_{I,W}$$

is a $\overline{\mathbb{Q}_\ell}$-linear functor from the category of representations of $\widehat{G}^I$ to the category of representations of $\mathrm{Gal}(\overline{F}/F)^I$.
This means that for any morphism

$$u : W \to W'$$

of representations of $\widehat{G}^I$, we have a morphism

$$\mathcal{H}(u) : H_{I,W} \to H_{I,W'}$$

of representations of $\mathrm{Gal}(\overline{F}/F)^I$.

b) Fusion for the $H_{I,W}$

Fusion can be associated to any map $\zeta : I \to J$ but we consider here only the case where J is a singleton, which we denote by $\{0\}$.

For any representation W of $\widehat{G}^I$, we have a fusion isomorphism, functorial in W,

$$H_{I,W} \xrightarrow{\sim} H_{\{0\},W_{\mathrm{diag}}}$$

where W_{diag} denotes the representation of $\widehat{G}$ on W obtained by composition with the diagonal morphism $\widehat{G} \to \widehat{G}^I$.

Two examples of the fusion isomorphism of the previous slide.

- If W_1 and W_2 are two representations of $\widehat{G}$, we have the fusion isomorphism

$$H_{\{1,2\},W_1\boxtimes W_2} \xrightarrow{\sim} H_{\{0\},W_1\otimes W_2}$$

  associated to the obvious map $\{1,2\} \to \{0\}$. We note the difference between $W_1 \boxtimes W_2$ which is a representation of $(\widehat{G})^2$ and $W_1 \otimes W_2$ which is a representation of $\widehat{G}$.

- We have the fusion isomorphism

$$H_{\emptyset,\mathbf{1}} \xrightarrow{\sim} H_{\{0\},\mathbf{1}}$$

  associated to the obvious map $\emptyset \to \{0\}$ (the idea is that $H_{\emptyset,\mathbf{1}}$, *resp.* $H_{\{0\},\mathbf{1}}$ is the cohomology of the stack of shtukas without legs, *resp.* with an inactive leg and that they are identical). Thus $\mathfrak{Aut} = H_{\{0\},\mathbf{1}}$ and we will use this equality in the next slide.

Construction of the algebra $\mathcal{B}$ of excursion operators.

For any algebraic function f on $\widehat{G}\backslash\widehat{G}^I/\widehat{G}$ we can find a representation W of $\widehat{G}^I$ and $x \in W$ and $\xi \in W^*$ invariant by the diagonal action of $\widehat{G}$ such that

$$f((g_i)_{i\in I}) = \langle \xi, (g_i)_{i\in I} \cdot x \rangle. \tag{0.2}$$

Let $(\gamma_i)_{i\in I} \in (\mathrm{Gal}(\overline{F}/F))^I$. The excursion operator $S_{I,f,(\gamma_i)_{i\in I}}$ of $H_{\{0\},\mathbf{1}} = \mathfrak{Aut}$ is defined as the composition

$$H_{\{0\},\mathbf{1}} \xrightarrow{\mathcal{H}(x)} H_{\{0\},W_{\mathrm{diag}}} \overset{\text{fusion}}{\xrightarrow{\sim}} H_{I,W} \xrightarrow{(\gamma_i)_{i\in I}} H_{I,W} \overset{\text{fusion}}{\xrightarrow{\sim}} H_{\{0\},W_{\mathrm{diag}}} \xrightarrow{\mathcal{H}(\xi)} H_{\{0\},\mathbf{1}}$$

where W_{diag} is the diagonal representation of $\widehat{G}$ on W, and $x : \mathbf{1} \to W_{\mathrm{diag}}$ and $\xi : W_{\mathrm{diag}} \to \mathbf{1}$ are considered here as morphisms of representations of $\widehat{G}$.
We show easily that the construction above does not depend on the choice of W, x, ξ satisfying (0.2).

Construction of the decomposition of the theorem.

Thanks to the properties of the $H_{I,W}$ explained in the previous slides we show that

1) the algebra $\mathcal{B}$ of endomorphisms of $H_{\{0\},\mathbf{1}} = \mathfrak{Aut}$ generated by the $S_{I,f,(\gamma_i)_{i\in I}}$ when I, f and $(\gamma_i)_{i\in I}$ vary is commutative and the $S_{I,f,(\gamma_i)_{i\in I}}$ satisfy some natural relations,

2) for any character ν of $\mathcal{B}$ there is a unique Langlands parameter σ such that for any I, f and $(\gamma_i)_{i\in I}$,

$$\nu(S_{I,f,(\gamma_i)_{i\in I}}) = f((\sigma(\gamma_i))_{i\in I}).$$

Since $\mathcal{B}$ is commutative we have a canonical spectral decomposition $H_{\{0\},\mathbf{1}} = \bigoplus_\nu \mathfrak{H}_\nu$ where the sum is taken over characters ν of $\mathcal{B}$ (in other words $\mathfrak{H}_\nu$ is a generalized eigenspace for the elements of $\mathcal{B}$). Associating to ν a unique Langlands parameter σ as in 2) we deduce the decomposition $H_{\{0\},\mathbf{1}} = \bigoplus_\sigma \mathfrak{H}_\sigma$ we wanted to construct.

Compatibility with the Satake isomorphism and end of the proof of the theorem.

The unramified Hecke operators are particular cases of excursion operators. Indeed let V be an irreducible representation of $\widehat{G}$. We take

$$I = \{1, 2\} \text{ and } f : (g_1, g_2) \mapsto \mathrm{Tr}_V(g_1 g_2^{-1}) \text{ as a function on } \widehat{G}\backslash\widehat{G}^I/\widehat{G}.$$

By a geometric argument (computing the intersection of algebraic cycles in the stack of shtukas) we show that for any closed point v,

$$T_{V,v} = S_{\{1,2\},f,(\mathrm{Frob}_v,1)}.$$

This equality plays an important role in technical arguments, and it justifies the compatibility of the decomposition with Satake isomorphism at closed points v of X.

The theorem is proven. Now I explain an heuristics which goes farther and unveils, a posteriori, why the construction of excursion operators worked.

The Arthur-Kottwitz heuristics for the $H_{I,W}$ is that for every σ there exists a $\overline{\mathbb{Q}_\ell}$-linear representation $\mathfrak{A}_\sigma$ of its centralizer $S_\sigma \subset \widehat{G}$, so that

$$H_{I,W} \stackrel{?}{=} \bigoplus_\sigma \left(\mathfrak{A}_\sigma \otimes W_{\sigma^I}\right)^{S_\sigma}$$

where S_σ acts diagonally and W_{σ^I} is the representation of $\mathrm{Gal}(\overline{F}/F)^I$ obtained by composition of the representation W of $\widehat{G}^I$ with the morphism $\sigma^I : \mathrm{Gal}(\overline{F}/F)^I \to \widehat{G}^I$. Taking $I = \emptyset$ and $W = \mathbf{1}$, $H_{\emptyset,\mathbf{1}} \stackrel{?}{=} \bigoplus_\sigma \left(\mathfrak{A}_\sigma\right)^{S_\sigma}$ should be the decomposition $H_{\emptyset,\mathbf{1}} = \bigoplus_\sigma \mathfrak{H}_\sigma$ of the theorem. Thus in this heuristics the excursion operator $S_{I,f,(\gamma_i)_{i\in I}}$ should act on the subspace $\mathfrak{H}_\sigma = \left(\mathfrak{A}_\sigma\right)^{S_\sigma}$ of $H_{\emptyset,\mathbf{1}} \simeq H_{\{0\},\mathbf{1}}$ by the composition

$$\left(\mathfrak{A}_\sigma \otimes \mathbf{1}\right)^{S_\sigma} \xrightarrow{\mathrm{Id}_{\mathfrak{A}_\sigma} \otimes x} \left(\mathfrak{A}_\sigma \otimes W\right)^{S_\sigma} \xrightarrow{\mathrm{Id}_{\mathfrak{A}_\sigma} \otimes (\sigma(\gamma_i))_{i\in I}} \left(\mathfrak{A}_\sigma \otimes W\right)^{S_\sigma} \xrightarrow{\mathrm{Id}_{\mathfrak{A}_\sigma} \otimes \xi} \left(\mathfrak{A}_\sigma \otimes \mathbf{1}\right)^{S_\sigma}$$

i.e. by multiplication by the scalar $\langle \xi, (\sigma(\gamma_i))_{i\in I} \cdot x\rangle = f((\sigma(\gamma_i))_{i\in I})$ and $\mathfrak{H}_\sigma$ should be the eigenspace of the excursion operators for this system of eigenvalues.

A construction proposed by Drinfeld.

We do not know the previous heuristics. For example we only know that $\mathfrak{H}_\sigma$ is a generalized eigenspace for the excursion operators (there could perhaps be nilpotents in the algebra $\mathcal{B}$).
Nevertheless, thanks to an idea of Drinfeld we can even obtain something close to the Arthur-Kottwitz heuristics. Let Reg be the left regular representation of $\widehat{G}$ (i.e. the action by left translation of $\widehat{G}$ on the vector space of all algebraic functions on $\widehat{G}$). We can endow $H_{\{0\},\mathrm{Reg}}$ with

a) a structure of $\mathcal{O}$-module on the "space" $\mathcal{S}$ of morphisms $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}$,

b) an action of $\widehat{G}$ compatible with conjugation by $\widehat{G}$ on $\mathcal{S}$.

This gives rise to a $\mathcal{O}$-module on the "stack" $\mathcal{S}/\widehat{G}$ of Langlands parameters and $\mathfrak{A}_\sigma$ should be the fiber of this $\mathcal{O}$-module at σ. It would be equipped with an action of the centralizer S_σ, which is the automorphism group of σ in $\mathcal{S}/\widehat{G}$.
Xinwen Zhu and I prove this works over elliptic σ (which means that S_σ is finite).

A joint work with Alain Genestier.

In the main theorem the canonical decomposition

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_{\sigma} \mathfrak{H}_\sigma, \tag{0.3}$$

is preserved by all Hecke operators, including ramified Hecke operators (not defined in this talk) at closed points v in N.

The theorem gives the compatibility with the Satake isomorphism at closed points in $X \setminus N$ but does not say how the action on $\mathfrak{H}_\sigma$ of ramified Hecke operators at closed points v in N is related to σ.

In a joint work with Alain Genestier, we construct a local parameterization up to semisimplification and show a local-global compatibility at all closed points.

It implies that in the decomposition above, for any closed point $v \in N$, the semisimplification of $\sigma\big|_{\mathrm{Gal}(\overline{F_v}/F_v)}$ depends only on the character by which the center of the algebra of ramified Hecke operators at v acts on $\mathfrak{H}_\sigma$.

Some open questions.

1) In this work we construct a decomposition $\mathfrak{Aut} = \bigoplus_\sigma \mathfrak{H}_\sigma$ but we are not able to compute multiplicities of the $\mathfrak{H}_\sigma$ as modules over the Hecke algebras (we are not even able to show that they are nonzero). Arthur has conjectured a formula for them. The trace formula methods allow to prove some cases (Drinfeld, Laumon, Laurent Lafforgue, Ngo Bao Chau, Lau, Ngo Dac Tuan and the work of Arthur for classical groups). Alain Genestier and I plan to use the methods explained here to study the internal structure of the local L-packets and the structure of the multiplicity formulas.

2) We hope that all Langlands parameters σ which appear in this decomposition come from elliptic Arthur parameters. This would imply the Ramanujan-Petersson conjecture for all reductive groups over function fields. We even hope that there is a decomposition of the vector space of all discrete automorphic forms indexed by elliptic Arthur parameters.

3) We hope that the decomposition

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma$$

is defined over $\overline{\mathbb{Q}}$ (instead of $\overline{\mathbb{Q}_\ell}$) and is independent on ℓ and on the embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}_\ell}$. The question makes sense because in a recent article Drinfeld defines the set of Langlands parameters σ independently of ℓ.

We could prove this is true if we knew how to construct the excursion operators in a motivic way (then the σ would be motivic Langlands parameters).

Grothendieck motives form a $\overline{\mathbb{Q}}$-linear category and unite all ℓ-adic cohomologies for different ℓ : a motive is a piece of "universal cohomology" of a variety.

Related works (it is not possible to quote all of them).

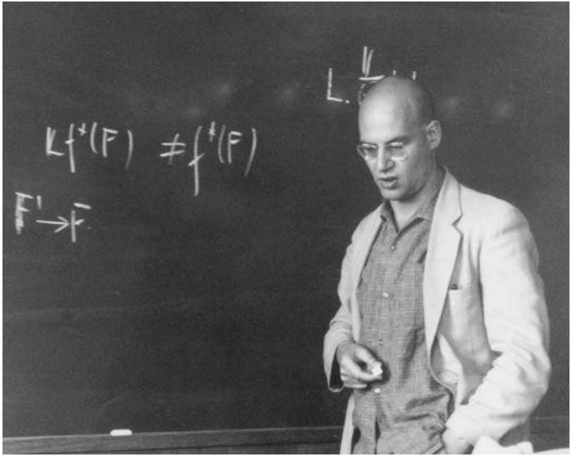
1) The work of Böckle, Harris, Khare, and Thorne on potential automorphy and the work of Zhiwei Yun and Wei Zhang on shtukas and L-functions were explained at this conference. A work of Liang Xiao and Xinwen Zhu clarified in particular the Eichler-Shimura relations (a technical tool not explained in this talk).

2) Finkelberg, Lysenko and Gaitsgory studied the metaplectic case of the ℓ-adic geometric Langlands program. In particular the metaplectic variant of the geometric Satake equivalence is used to extend the main theorem of this talk to the metaplectic case.

3) An important work of Gaitsgory in the geometric Langlands program for D-modules shows that, when X is a curve over $\mathbb{C}$, the ∞-category $\text{D-mod}(\text{Bun}_G)$ of D-modules over Bun_G admits a spectral decomposition along the stack $\text{LocSys}_{\widehat{G}}$ of $\widehat{G}$-local systems over X. Arinkin and Gaitsgory defined a ∞-category of $\mathcal{O}$-modules over $\text{LocSys}_{\widehat{G}}$ to which $\text{D-mod}(\text{Bun}_G)$ should be equivalent. Many progresses were made also in the ramified situation and the local geometric Langlands program, and in the quantum geometric Langlands program.

4) Gaitsgory and Lurie used ideas related to fusion to prove Weil's conjecture on Tamagawa numbers over function fields.

Homage to Alexander Grothendieck (1928-2014).



As other works in algebraic geometry, this one relies on ideas of Grothendieck : functorial definition of schemes and stacks, tannakian formalism, Quot construction of Bun_G, étale cohomology. His vision of topos and motives already had tremendous consequences and others are certainly yet to come. He also had a strong influence outside of his school, as testified by the rise of higher categories and the work of Beilinson, Drinfeld, Gaitsgory, Kontsevich, Lurie, Voevodsky (who, sadly, passed away recently) and many others. He changed not only mathematics, but also the way we think about it.